STABILITY AND INSTABILITY OF THE EINSTEIN-LICHNEROWICZ CONSTRAINT SYSTEM

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ABSTRACT. We investigate the relevance of the conformal method by investigating stability issues for the Einstein-Lichnerowicz conformal constraint system in a nonlinear scalar-field setting. We prove the stability of the system with respect to arbitrary perturbations of generic focusing physics data on closed locally conformally flat manifolds, in any dimension. We also show that our stability result is sharp by constructing explicit instability examples when its assumptions are not satisfied. Our results apply to a more general class of constraint-like systems.

1. Introduction

The constraint equations arise in General Relativity, in the analysis of the initial-value problem for the Einstein equations. Given a n-manifold M, $n \geq 3$, and a potential V – a smooth function in $\mathbb{R}$ – in a nonlinear scalar-field setting the constraint equations write as follows:

$$\begin{cases} R(\tilde{g}) + (\mathrm{tr}_{\tilde{g}}\tilde{K})^2 - \|\tilde{K}\|_{\tilde{g}}^2 = \tilde{\pi}^2 + |\nabla\tilde{\psi}|_{\tilde{g}}^2 + 2V(\tilde{\psi}), \\ \nabla^j\tilde{K}_{ij} - \nabla_i(\mathrm{tr}_{\tilde{g}}\tilde{K}) = \tilde{\pi}\nabla_i\tilde{\psi}. \end{cases} \tag{$\mathcal{C}_-$}$$

The unknowns of $(\mathcal{C}_-)$ are the Riemannian metric $\tilde{g}$ in M, $\tilde{K}$ a $(2,0)$-symmetric tensor field and $\tilde{\psi}$ and $\tilde{\pi}$ two functions in M, representing the scalar-field and its future-directed temporal derivative. Also, $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$ and $R(\tilde{g})$ is its scalar curvature. A solution $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ of $(\mathcal{C}_-)$ is called an *initial data set*. This terminology has its roots in the celebrated well-posedness results of Choquet-Bruhat [20] and Choquet-Bruhat-Geroch [9], that assert that every initial data set $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ possesses a unique maximal globally hyperbolic spacetime development $(\mathcal{M}, h, \Psi)$ satisfying the Einstein scalar-field equations. The system $(\mathcal{C}_-)$ is therefore of paramount importance in General Relativity since it is a necessary and sufficient condition for the resolution of the Einstein equations. Furthermore, addressing the resolution of $(\mathcal{C}_-)$ exactly amounts to determining the initial data set of the evolution.

One of the most successful techniques developed to overcome the underdetermination of $(\mathcal{C}_-)$ is the conformal method. Initiated by Lichnerowicz [32] and improved by Choquet-Bruhat and York [10], it aims at turning $(\mathcal{C}_-)$ into a determined system by parametrizing the unknowns in terms of prescribed background physics data. We will assume from now on that M is a closed manifold – that is, compact without boundary – endowed with a reference Riemannian metric g. We let a potential V and $(\psi, \pi, \tau, \sigma)$ be fixed physics data in M, where ψ, π, τ are functions and σ is a traceless and divergence-free $(2,0)$ symmetric tensor in M. We

look for solutions of $(\mathcal{C}_-)$ under the following parametrization:

$$(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi}) = \left(\varphi^{\frac{4}{n-2}} g, \frac{\tau}{n}\varphi^{\frac{4}{n-2}} g + \varphi^{-2}(\sigma + \mathcal{L}_g W), \psi, \varphi^{-\frac{2n}{n-2}}\pi\right), \tag{1.1}$$

where $\varphi > 0$ is a positive function in M, W is a field of 1-forms and

$$\mathcal{L}_g W_{ij} = W_{i,j} + W_{j,i} - \frac{2}{n}\left(\mathrm{div}_g W\right) g_{ij} \tag{1.2}$$

is the conformal Killing derivative of W. As easily checked, $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ given by (1.1) solves $(\mathcal{C}_-)$ if and only if (φ, W) solves the following *Einstein-Lichnerowicz conformal constraint system of physics data $D = (\psi, \pi, \tau, \sigma)$ and V*. Namely:

$$\begin{cases} \dfrac{4(n-1)}{n-2}\triangle_g \varphi + \mathcal{R}_\psi \varphi = \mathcal{B}_{\tau,\psi,V}\varphi^{2^*-1} + \dfrac{\left(|\sigma + \mathcal{L}_g W|_g^2 + \pi^2\right)}{\varphi^{2^*+1}} \ , \\ \overrightarrow{\triangle}_g W = -\dfrac{n-1}{n}\varphi^{2^*}\nabla\tau - \pi\nabla\psi \ . \end{cases} \tag{C_D}$$

In (C_D), the coefficients express in terms of the physics data as:

$$\mathcal{R}_\psi = R(g) - |\nabla\psi|_g^2 \ , \ \mathcal{B}_{\tau,\psi,V} = 2V(\psi) - \frac{n-1}{n}\tau^2. \tag{1.3}$$

Also, $2^* = \frac{2n}{n-2}$ is the critical exponent for the embedding of the Sobolev space $H^1(M)$ into Lebesgue spaces, $\triangle_g = -\mathrm{div}_g(\nabla \cdot)$ is the Laplace-Beltrami operator and $\overrightarrow{\triangle}_g$ is the Lamé operator acting on 1-forms:

$$\overrightarrow{\triangle}_g W = -\mathrm{div}_g(\mathcal{L}_g W).$$

Since M is closed, the operator $\overrightarrow{\triangle}_g$ is elliptic and self-adjoint on TM and possesses the usual regularizing fetaures of elliptic operators. Fields of 1-forms W satisfying $\mathcal{L}_g W = 0$ in M will be called conformal Killing 1-forms, see (2.3) below. System (C_D) is in particular invariant up to the addition of any conformal Killing 1-form to W. In the following we shall therefore always considers solutions (φ, W) of (C_D) *up to conformal Killing 1-forms*, that is we shall assume that W is L^2-orthogonal to the space of such conformal Killing 1-forms.

The conformal method, starting from given physics data, generates initial data sets that consequently provide us with space-time developments. The whole procedure can be summed up in the following 3-steps construction, that we shall call the Choquet-Bruhat-Geroch-Lichnerowicz (CBGL) formalism:

$$\begin{array}{ll} \text{Freely chosen physics data } (\psi, \pi, \tau, \sigma) \text{ and } V & \\ \quad\big\downarrow & \text{Step 1: Solving } (C_D) \\ \text{Solution(s) } (\varphi, W) \text{ of } (C_D) & \\ \quad\big\downarrow & \text{Step 2: Parametrization (1.1)} \\ \text{Initial data set(s) } \left(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi}\right) & \\ \quad\big\downarrow & \text{Step 3: Existence result [9]} \\ \text{Maximal space-time development(s) } (\mathcal{M}, h, \Psi) & \end{array}$$

Space-times obtained via the latter construction capture the features of the specific initial data sets $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ obtained through the conformal method. Investigating

the validity of the conformal method therefore reduces to the investigation of the physical relevance of the CBGL formalism. We address it here through the following fundamental question:

Question: *is the CBGL formalism robust with respect to the initial choice of the physics data* $(\psi, \pi, \tau, \sigma)$ *and* V *of the conformal method?*

More specifically, we do not expect minor perturbations of the physics data $(\psi, \pi, \tau, \sigma)$ and V to create dramatic changes in the geometry of the resulting spacetimes. In the 3-steps CBGL construction, once an initial data set $(\tilde{g}, \tilde{K}, \tilde{\psi}, \tilde{\pi})$ is given, continuity for Step 3 is ensured by the notion of Cauchy stability (see Ringström, [36]). Clearly, Step 2 is continuous. Hence, the crucial point lies in the proof of the continuity of Step 1. Proving the robustness of the CBGL formalism and of the conformal method therefore boils down to proving the stability of system (C_D), that is the continuous dependence of the set of solutions of (C_D) on the choice of the physics data $(\psi, \pi, \tau, \sigma)$ and V.

We introduce the following terminology: we shall say that physics data V and $(\psi, \pi, \tau, \sigma)$ are

$$\textit{focusing} \text{ if } \mathcal{B}_{\tau,\psi,V} > 0 \text{ in } M \quad \text{and} \quad \textit{defocusing} \text{ if } \mathcal{B}_{\tau,\psi,V} \le 0 \text{ in } M, \tag{1.4}$$

where $\mathcal{B}_{\tau,\psi,V}$ is as in (1.3).

In the present work we establish the stability of system (C_D) in strong topologies with respect to arbitrary perturbations of focusing physics data on locally conformally flat manifolds, in any dimension. It is the content of our main result:

Theorem 1.1. *Let* (M, g) *be a closed locally conformally flat Riemannian manifold of dimension* $n \ge 3$. *Consider a smooth potential* V *and smooth physics data* $D = (\psi, \pi, \tau, \sigma)$. *Assume that the data are focusing as in* (1.4) *and that* $\pi \not\equiv 0$. *If* $n \ge 6$, *assume in addition that* τ *and* ψ *have no common critical points in* M. *Let* $(V_\alpha)_\alpha$ *and* $(D_\alpha)_\alpha$, $D_\alpha = (\psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)_\alpha$ *be sequences of potentials and of physics data converging respectively to* V *and* D *in the following topology:*

$$\|V_\alpha - V\|_{C^2} + \|\tau_\alpha - \tau\|_{C^3} + \|\psi_\alpha - \psi\|_{C^2} + \|\pi_\alpha - \pi\|_{C^0} + \|\sigma_\alpha - \sigma\|_{C^0} \underset{\alpha \to +\infty}{\longrightarrow} 0. \tag{1.5}$$

Consider $(\varphi_\alpha, W_\alpha)_\alpha$, $\varphi_\alpha > 0$, *a sequence of solutions of the Einstein-Lichnerowicz constraints system of physics data* D_α *and* V_α*:*

$$\begin{cases} \dfrac{4(n-1)}{n-2} \triangle_g \varphi_\alpha + \mathcal{R}_{\psi_\alpha} \varphi_\alpha = \mathcal{B}_{\tau_\alpha,\psi_\alpha,V_\alpha} \varphi_\alpha^{2^*-1} + \dfrac{\pi_\alpha^2 + |\sigma_\alpha + \mathcal{L}_g W_\alpha|_g^2}{\varphi_\alpha^{2^*+1}} \,, \\ \overrightarrow{\triangle}_g W_\alpha = -\dfrac{n-1}{n} \varphi_\alpha^{2^*} \nabla \tau_\alpha - \pi_\alpha \nabla \psi_\alpha \,, \end{cases} \tag{C_{D_α}}$$

where the coefficients express as in (1.3). *Then, up to a subsequence and up to conformal Killing* 1*-forms, the sequence* $(\varphi_\alpha, W_\alpha)_\alpha$ *converges in* $C^{1,\eta}(M)$, *for any* $0 < \eta < 1$, *to some solution* (φ_0, W_0), $\varphi_0 > 0$, *of the limiting Einstein-Lichnerowicz constraints system of equations* (C_D).

Theorem 1.1 is also a compactness result. It establishes in particular that sequences of solutions of (C_D) for perturbations of given physics data do not blow-up. Although not explicitly stated in Theorem 1.1, the result still holds true if we also allow perturbations of the geometry of M in the locally conformally flat category in

strong topologies. The consequence of Theorem 1.1 in terms of the CBGL formalism is as follows:

Corollary 1.2. *The CBGL formalism is stable with respect to the choice of generic focusing initial data* $(\psi, \pi, \tau, \sigma)$ *and* V *in any locally conformally flat geometry in* M*.*

The focusing case investigated here and defined in (1.4) covers the general physical setting where nontrivial nongravitational data are considered. Among the prominent cases allowed one finds for instance the positive cosmological constant case or the nontrivial Klein-Gordon field case. Note that in small dimensions Theorem 1.1 only requires the non-staticness assumption $\pi \not\equiv 0$ while in dimensions $n \geq 6$ stability is ensured by a higher order stationarity condition on the scalar-field ψ and the mean curvature τ. These assumptions are in particular generic on the set of all focusing physics data $(\psi, \pi, \tau, \sigma)$ and V.

Obviously, solutions of (C_D) exist, at least in some cases. Partial existence results for the vacuum defocusing case were obtained in Isenberg [29], Holst-Nagy-Tsogtgerel [28], Maxwell [33] and Dahl-Gicquaud-Humbert [13]. Only recently the more involved focusing case, which is the case of interest in Theorem 1.1, has been addressed: we refer to Hebey-Pacard-Pollack [25], Premoselli [35, 34] and Holst-Meier [27]. Stability results for system (C_D) had been obtained in specific cases, see Druet-Hebey [15], Premoselli [34] and Druet-Premoselli [19]. Note also that, unlike all the known existence results, Theorem 1.1 requires no smallness assumptions on the physics data and allows, in full generality, the manifold M to possess non-trivial conformal Killing 1-forms in M. As a remark, for geometric nonlinear critical elliptic equations, low-regularity perturbations of the metric and/or of the coefficients may lead to the existence of blowing-up sequences of solutions. See for instance Berti-Malchiodi [3], Druet-Laurain [18] and Druet-Hebey-Laurain [16] for examples of such phenomena for nonlinear stationary Schrödinger equations. In dimensions 3, the convergence of $(V_\alpha)_\alpha$ and $(D_\alpha)_\alpha$ can be lowered from C^2 to C^1.

Note also that, since system (C_D) is elliptic, as the regularity of the convergence of $(D_\alpha)_\alpha$ to D and of $(V_\alpha)_\alpha$ to V increases, the regularity of convergence of the solutions increases accordingly. For physics data converging in $C^\infty(M)$ the convergence of solutions in Theorem 1.1 holds in $C^\infty(M)$ too.

Let us point out one important fact. We are investigating here a specific notion of elliptic stability (see Hebey [24] for a reference in book form), which turns out to be particularly well-suited for the conformal method and the CBGL construction. As already noticed, in our setting, the notion precisely measures the robustness of the CBGL construction with respect to the choice of the upstream physics data of the problem. There are other very natural (and more historical) notions of stability which arise when investigating the stability of specific solutions of the Einstein equations. There is a huge litterature in this deep direction. Without pretending to be exhaustive, we mention the groundbreaking global nonlinear stability of the Minkowski space-time by Christodoulou-Klainerman [11] (see also Klainerman-Nicolò [31] or Lindblad-Rodnianski [38]) and refer to Dafermos-Rodnianski [12] and the references therein for a detailed account of the existing work on the problem of the stability of black holes.

The proof of Theorem 1.1 goes through the proof of a stability result for a general class of constraint-like systems, see (2.1) and Theorem 2.1 in section 2 below. The proof proceeds by contradiction and requires an involved asymptotic analysis of blowing-up sequences of solutions of (2.1) around concentration points. In Section 3 we isolate the regions of M where loss of compactness may occur and we show that concentration points do not actually appear in M. The proof of this fact – that concludes the proof of Theorem 1.1 – crucially relies on an accurate pointwise asymptotic description of the behavior of blowing-up sequences of solutions around concentration points. Such an asymptotic description is by far the core of the analysis in this paper and, for the sake of clarity, is postponed to Section 4. Unlike in the fully decoupled case (where $\nabla \tau \equiv 0$), treated in Druet-Hebey [15] and Premoselli [34], obtaining a sharp pointwise description of blowing-up sequences of (C_D) in full generality as we do here requires an involved simultaneous analysis of the defects of compactness that occur in each of the two equations of (C_D) and of their interactions. We perform this analysis in Section 4. Finally, section 5 is aimed at showing that the assumptions of Theorem 2.1 below are sharp and is concerned with the construction of blowing-up sequences of solutions of constraint-like systems of equations, and sections 6 to 8 gather some technical results used throughout the paper.

2. A PDE framework

We investigate the Einstein-Lichnerowicz conformal constraint system (C_D) in an elliptic PDE framework and prove a general stability result for an extended class of constraint-like systems. We consider (M, g) a closed locally conformally flat manifold of dimension $n \geq 3$ and consider a sequence $(u_\alpha, W_\alpha)_\alpha$ of solutions of:

$$\begin{cases} \triangle_g u_\alpha + h_\alpha u_\alpha = f_\alpha u_\alpha^{2^*-1} + \dfrac{a_\alpha(W_\alpha)}{u_\alpha^{2^*+1}}, \\ \overrightarrow{\triangle}_g W_\alpha = u_\alpha^{2^*} X_\alpha + Y_\alpha, \end{cases} \tag{2.1}$$

with $u_\alpha > 0$ in M, where

$$a_\alpha(W_\alpha) = b_\alpha + |U_\alpha + \mathcal{L}_g W_\alpha|_g^2,$$

$h_\alpha, f_\alpha, b_\alpha$ are smooth functions in M, X_α, Y_α are smooth 1-forms in M and U_α is a smooth symmetric $(2,0)$ tensor field in M. Here, as before, $2^* = \frac{2n}{n-2}$, $\triangle_g = -\mathrm{div}_g(\nabla \cdot)$,

$$\mathcal{L}_g W_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n}(\mathrm{div}_g W) g_{ij}$$

and $\overrightarrow{\triangle}_g = -\mathrm{div}_g(\mathcal{L}_g \cdot)$. We assume that there holds

$$(h_\alpha, f_\alpha, b_\alpha, U_\alpha, X_\alpha, Y_\alpha)_\alpha \to (h_0, f_0, b_0, U_0, X_0, Y_0) \tag{2.2}$$

as $\alpha \to \infty$, where all the convergences take place in $C^0(M)$ except the convergence of $(f_\alpha)_\alpha$ and $(X_\alpha)_\alpha$ to f_0 and X_0 which take place in $C^2(M)$. We assume in addition that $\triangle_g + h_0$ is coercive and that $f_0 > 0$.

We let K_g denote the set of conformal Killing 1-forms in M:

$$K_g = \{W \in H^1(M) \text{ st } \mathcal{L}_g W = 0\}, \tag{2.3}$$

where $H^1(M)$ denotes the usual Sobolev space on 1-forms. Since the only quantity depending on W_α that appears in system (2.1) is $\mathcal{L}_g W_\alpha$ the system is invariant

under the addition to W_α of elements of K_g. In the following we may therefore consider sequences of solutions $(u_\alpha, W_\alpha)_\alpha$ *up to conformal Killing* 1*-forms*, that is we may assume that W_α is orthogonal to K_g for the L^2-scalar product.

Our main result is an *a priori* boundedness result:

Theorem 2.1. *Let $(h_\alpha, f_\alpha, b_\alpha, U_\alpha, X_\alpha, Y_\alpha)_\alpha$ be a sequence of coefficients, converging to some limiting coefficients $(h_0, f_0, b_0, U_0, X_0, Y_0)$, and satisfying the convergence conditions of* (2.2)*. Assume that $\triangle_g + h_0$ is coercive and that $f_0 > 0$ in M. If $3 \le n \le 5$, assume that either $b_0 \not\equiv 0$ or $|X_0|_g > 0$ in M. If $n \ge 6$, assume that X_0 and ∇f_0 have no common zero or, if they do, assume that there holds at these zeroes:*

$$h_0 < \frac{n-2}{4(n-1)} R(g) - C(n) f_0^{-1} \triangle_g f_0, \tag{2.4}$$

where $C(n)$ denotes some positive constant only depending on n explicitly given in (4.120) *below, and $R(g)$ is the scalar curvature of g. Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of* (2.1)*. Let $0 < \eta < 1$. There exists then a positive constant $C(\eta)$ that does not depend on α such that, up to a subsequence and up to conformal Killing* 1*-forms,*

$$\|u_\alpha\|_{L^\infty(M)} + \|W_\alpha\|_{C^{1,\eta}(M)} \le C(\eta),$$

for all α.

The limiting system associated to (2.1)–(2.2) is:

$$\begin{cases} \triangle_g u_0 + h_0 u_0 = f_0 u_0^{2^*-1} + \left(b_0 + |U_0 + \mathcal{L}_g W_0|_g^2\right) u_0^{-2^*-1}, \\ \overrightarrow{\triangle}_g W_0 = u_0^{2^*} X_0 + Y_0. \end{cases} \tag{2.5}$$

As a consequence of Theorem 2.1 we have the following compactness result:

Corollary 2.2. *Under the assumptions of Theorem 2.1 there holds:*

(1) *either $\|u_\alpha\|_{L^\infty(M)} \to 0$ as $\alpha \to \infty$, in which case $b_0 \equiv 0$, $div_g U_0 = Y_0$ in M and, up to a subsequence and up to conformal Killing* 1*-forms, $W_\alpha \to W_0$ in $C^{1,\eta}(M)$ for any $0 < \eta < 1$, where $\overrightarrow{\triangle}_g W_0 = Y_0$ in M,*
(2) *or $\|u_\alpha\|_{L^\infty(M)} \not\to 0$ as $\alpha \to \infty$. In this case, up to a subsequence and up to conformal Killing* 1*-forms, $(u_\alpha, W_\alpha) \to (u_0, W_0)$ in $C^{1,\eta}(M)$ for any $0 < \eta < 1$ with $u_0 > 0$ in M, where (u_0, W_0) is a solution of the limiting system* (2.5)*.*

In both cases (u_α, W_α) converges to a solution of the limiting system, even if it is somewhat degenerate in the first case.

Proof. Assume that the first alternative holds. By standard elliptic theory, up to conformal Killing 1-forms, $W_\alpha \to W_0$ in $C^{1,\eta}(M)$ where $\overrightarrow{\triangle}_g W_0 = Y_0$. Let G_α be the Green function of the operator $\triangle_g + h_\alpha$ in M. It is uniformly positive thanks to (2.2) and since $\triangle_g + h_0$ is coercive (see Robert [37]). A Green's formula on the scalar equation gives, for some positive constant C:

$$\|u_\alpha\|_{L^\infty(M)}^{2^*+2} \ge \frac{1}{C} \int_M \left(b_\alpha + |U_\alpha + \mathcal{L}_g W_\alpha|_g^2\right) dv_g$$

which shows that $b_0 \equiv 0$ and $U_\alpha + \mathcal{L}_g W_\alpha \to 0$ in $L^2(M)$. Passing to the limit we obtain $\mathrm{div}_g U_0 = Y_0$.

In the second case, if $\|u_\alpha\|_{L^\infty(M)} \not\to 0$, the Harnack inequality as stated in Section 6 shows that $\inf_M u_\alpha \ge \frac{1}{C}$ for some positive constant C. The $C^{1,\eta}(M)$-bounds on u_α and W_α therefore follow by standard elliptic theory in each equation. $\square$

Note that if the assumptions of Theorem 2.1 are satisfied the second alternative in Corollary 2.2 holds in dimensions $3 \le n \le 5$.

Compactness and stability results for elliptic PDEs have a long time history. A major result was the complete proof of the compactness of the Yamabe equation by Khuri-Marques-Schoen [30] together with its dimensional limitation by Brendle [6] and Brendle-Marques [7]. Note that an equation can be compact but unstable, that is, sensitive to changes of the parameters in the equation. General references on the stability of elliptic PDEs are by Druet [14], Druet-Hebey-Robert [17] and Hebey [24]. The specific case of the Einstein-Lichnerowicz equation is addressed in Druet-Hebey [15], Hebey-Veronelli [26] and Premoselli [34].

Theorem 2.1 leaves open the question of stability when its assumptions are not satisfied. The reason for this is that they are sharp. As shown in Section 5 below, whenever the assumptions of Theorem 2.1 are not satisfied we can construct blowing-up sequences of solutions of (2.1):

Theorem 2.3. *For any $n \ge 3$, when we contradict the assumptions $b_0 \not\equiv 0$ and* (2.4)*, there exist examples of smooth closed Riemannian n-manifolds (M, g) and of sequences of coefficients $(h_\alpha, f_\alpha, b_\alpha, U_\alpha, X_\alpha, Y_\alpha)_\alpha$ converging to some limiting coefficients $(h_0, f_0, b_0, U_0, X_0, Y_0)$ as in* (2.2) *with $U_0 \not\equiv 0$ and $X_0 \not\equiv 0$, and there exist sequences $(u_\alpha, W_\alpha)_\alpha$ of solutions of* (2.1) *such that $\sup_M u_\alpha \to +\infty$ as $\alpha \to +\infty$.*

The precise statements gathered in Theorem 2.3 are given in Section 5.

Theorem 2.1 highlights a dimensional hiatus in the behavior of (2.1). In dimensions $3 \le n \le 5$, the assumption $b_0 \not\equiv 0$ ensures that any sequence $(u_\alpha)_\alpha$ of solutions blows up with a (hypothetic) nonzero limit profile (compare with Proposition 5.1 where blowing-up sequences with zero limit profiles are constructed). Such an a priori property is crucially used in the blow-up analysis to control the radius of extension of the defects of compactness of the sequence $(u_\alpha, W_\alpha)_\alpha$. Conversely, when $n \ge 6$ or X_0 never vanishes, estimations of the extension radius are directly obtained in the analysis. Also, Theorem 2.1 requires (M, g) to be locally conformally flat. This assumption is crucial in our approach at this stage to get uniform estimates on the Green 1-forms of the operator $\overrightarrow{\triangle}_g$ with Neumann boundary conditions on small balls. If g is not locally conformally flat, the Kernel of $\mathcal{L}_g$ on small balls degenerates and this leads to a possible loss of compactness which is not quantifiable with the existing blow-up techniques.

3. Proof of Theorem 2.1

We let (M, g) be a closed locally conformally flat manifold of dimension $n \ge 3$ and we let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of (2.1) on (M, g) with $u_\alpha > 0$ in M. Assume that (2.2) holds. To prove Theorem 2.1 it is enough to show that there exists some positive constant C such that

$$\|u_\alpha\|_{L^\infty(M)} \le C.$$

Indeed, since the system (2.1) is invariant up to adding to W_α elements of (2.3), if we assume W_α to be orthogonal to K_g as in (2.3) there holds, by standard elliptic theory for the second equation (see Section 7), that $\|W_\alpha\|_{C^{1,\eta}(M)} \le C'$ for all $0 < \eta < 1$, for some positive constant C'. We proceed by contradiction and assume thus that there holds:

$$\sup_M u_\alpha \to +\infty \tag{3.1}$$

as $\alpha \to \infty$. In all of this section and in the remaining of the paper the letter C will always denote some positive constant, whose value may change from one line to another, but that will never depend on α.

Proposition 3.1. *Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of (2.1) such that (2.2) and (3.1) hold. There exists $N_\alpha \in \mathbb{N}^*$ and N_α points $(x_{1,\alpha}, \cdots, x_{N_\alpha,\alpha})$ of M satisfying, up to a subsequence:*

1. *$\nabla u_\alpha(x_{i,\alpha}) = 0$ for $1 \le i \le N_\alpha$,*
2. *$d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{n-2}{2}} u_\alpha(x_{i,\alpha}) \le 1$ for all $1 \le i, j \le N_\alpha$, $i \ne j$, and*
3. *there exists a positive constant C independent of α such that*
$$\left(\min_{1\le i\le N_\alpha} d_g(x_{i,\alpha}, x)\right)^n \left(u_\alpha(x)^{2^*} + |\mathcal{L}_g W_\alpha|_g(x)\right) \le C \tag{3.2}$$
for any $x \in M$.

Proof. Applying Lemma 1.1 in Druet-Hebey [15] we obtain that for any α there exists $N_\alpha \ge 1$ and N_α critical points $x_{1,\alpha}, \cdots, x_{N_\alpha,\alpha}$ of u_α satisfying:

$$d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{n-2}{2}} u_\alpha(x_{i,\alpha}) \ge 1 \quad \text{for } 1 \le i, j \le N_\alpha, i \ne j,$$

and

$$\left(\min_{1\le i\le N_\alpha} d_g(x_{i,\alpha}, x)\right)^{\frac{n-2}{2}} u_\alpha(x) \le 1 \tag{3.3}$$

for any critical point x of u_α. For any $x \in M$ let

$$\Psi_\alpha(x) = \left(\min_{1\le i\le N_\alpha} d_g(x_{i,\alpha}, x)\right)^n \left(u_\alpha(x)^{2^*} + |\mathcal{L}_g W_\alpha|_g(x)\right).$$

Assume by contradiction that (3.2) is false, and let $x_\alpha \in M$ be a sequence of points of M such that

$$\Psi_\alpha(x_\alpha) = \sup_M \Psi_\alpha \to +\infty \tag{3.4}$$

as $\alpha \to +\infty$. Define μ_α by:

$$\mu_\alpha^{-n} = u_\alpha(x_\alpha)^{2^*} + |\mathcal{L}_g W_\alpha|_g(x_\alpha). \tag{3.5}$$

By (3.4) there holds:

$$\frac{d_g(x_\alpha, \mathcal{S}_\alpha)}{\mu_\alpha} \to +\infty \tag{3.6}$$

where $\mathcal{S}_\alpha = \{x_{1,\alpha}, \cdots, x_{N_\alpha,\alpha}\}$ is constructed above and since M is compact, there holds

$$\mu_\alpha \to 0 \tag{3.7}$$

as $\alpha \to +\infty$. Since (M, g) is locally conformally flat we can let $\delta < i_g(M)$ and we can find a local chart $(B_{x_\alpha}(\delta), \Phi_\alpha)$ centered at x_α such that in this chart there holds

$$g_{ij} = \varphi_\alpha^{\frac{4}{n-2}} \xi_{ij}$$

for $1 \leq i,j \leq n$ and for some positive function φ_α in $\Phi_\alpha(B_{x_\alpha}(\delta))$ satisfying in addition:

$$\varphi_\alpha(0) = 1 \text{ and } \nabla\varphi_\alpha(0) = 0. \tag{3.8}$$

In addition to (3.8) we can choose the conformal factor φ_α to be bounded in the $C^k(M)$-topology for any $k \geq 0$ in $B_0(\delta)$. The operators $\triangle_g, \mathcal{L}_g$ and $\overrightarrow{\triangle}_g$ satisfy some conformal invariance formulae. Let v be a smooth function in M and X be a smooth 1-form in M. For any $x \in \Phi_\alpha(B_{x_\alpha}(\delta))$ we have that:

$$\triangle_\xi\left(\varphi_\alpha v \circ \Phi_\alpha^{-1}\right)(x) = \varphi_\alpha^{2^*-1}(x)\left[\triangle_g v(x) + \frac{n-2}{4(n-1)}R(g)v\right](\Phi_\alpha^{-1}(x)), \tag{3.9}$$

that

$$\varphi_\alpha^{\frac{4}{n-2}}\mathcal{L}_\xi\left(\varphi_\alpha^{-\frac{4}{n-2}}\left(\Phi_\alpha\right)_* X\right) = \left(\Phi_\alpha\right)_*\left(\mathcal{L}_g X\right), \tag{3.10}$$

and that for $1 \leq i \leq n$:

$$\begin{aligned}\overrightarrow{\triangle}_\xi\left(\varphi_\alpha^{-\frac{4}{n-2}}\left(\Phi_\alpha\right)_* X\right)_i - 2^*\xi^{kl}\partial_k\left(\ln\varphi_\alpha\right)\mathcal{L}_\xi\left(\varphi_\alpha^{-\frac{4}{n-2}}\left(\Phi_\alpha\right)_* X\right)_{li}& \\ = \left(\Phi_\alpha\right)_*\left(\overrightarrow{\triangle}_\xi X\right)_i&.\end{aligned} \tag{3.11}$$

We let, for any $x \in B_0(\frac{\delta'}{\mu_\alpha})$:

$$\begin{aligned}\hat{u}_\alpha(x) &= \mu_\alpha^{\frac{n-2}{2}}\varphi_\alpha(\mu_\alpha x)u_\alpha \circ \Phi_\alpha^{-1}(\mu_\alpha x),\\ \hat{W}_\alpha(x) &= \mu_\alpha^{n-1}\varphi_\alpha(\mu_\alpha x)^{-\frac{4}{n-2}}\left(\Phi_\alpha\right)_* W_\alpha(\mu_\alpha x),\end{aligned} \tag{3.12}$$

so that, using (3.9) and (3.11) the sequence $(\hat{u}_\alpha, \hat{W}_\alpha)$ satisfies the following system of equations in $B_0\left(\frac{\delta'}{\mu_\alpha}\right)$:

$$\begin{cases}\triangle_\xi\hat{u}_\alpha + \mu_\alpha^2\hat{h}_\alpha\hat{u}_\alpha = \hat{f}_\alpha\hat{u}_\alpha^{2^*-1} + \dfrac{\hat{a}_\alpha}{\hat{u}_\alpha^{2^*+1}},\\ \left(\overrightarrow{\triangle}_\xi\hat{W}_\alpha\right)_i = 2^*\mu_\alpha\xi^{kl}\partial_k(\ln\varphi_\alpha)(\mu_\alpha\cdot)\left(\mathcal{L}_\xi\hat{W}_\alpha\right)_{li} + \mu_\alpha\hat{u}_\alpha^{2^*}\left(\hat{X}_\alpha\right)_i + \mu_\alpha^{n+1}\left(\hat{Y}_\alpha\right)_i,\end{cases} \tag{3.13}$$

where we have let:

$$\begin{aligned}\hat{a}_\alpha(x) &= \mu_\alpha^{2n}\hat{b}_\alpha(x) + \left|\mu_\alpha^n\hat{U}_\alpha(x) + \varphi_\alpha(\mu_\alpha x)^{2^*}\mathcal{L}_\xi\hat{W}_\alpha\right|_\xi^2,\\ \hat{h}_\alpha(x) &= \varphi_\alpha(\mu_\alpha x)^{\frac{4}{n-2}}\left(h_\alpha \circ \Phi_\alpha^{-1} - \frac{n-2}{4(n-1)}R(g)\right)(\mu_\alpha x),\\ \hat{f}_\alpha(x) &= f_\alpha \circ \Phi_\alpha^{-1}(\mu_\alpha x),\\ \hat{b}_\alpha(x) &= \varphi_\alpha(\mu_\alpha x)^{2\cdot 2^*}b_\alpha \circ \Phi_\alpha^{-1}(\mu_\alpha x),\\ \hat{U}_\alpha(x) &= \varphi_\alpha(\mu_\alpha x)^2\left(\Phi_\alpha\right)_* U_\alpha(\mu_\alpha x),\\ \hat{X}_\alpha(x) &= \varphi_\alpha(\mu_\alpha x)^{-2^*}\left(\Phi_\alpha\right)_* X_\alpha(\mu_\alpha x) \text{ and}\\ \hat{Y}_\alpha(x) &= \left(\Phi_\alpha\right)_* Y_\alpha(\mu_\alpha x).\end{aligned} \tag{3.14}$$

By definition of μ_α in (3.5) there holds

$$\hat{u}_\alpha(0)^{2^*} + |\mathcal{L}_\xi\hat{W}_\alpha|_\xi(0) = 1 \tag{3.15}$$

and using (3.4) and (3.5) there holds, for any $R > 0$:

$$\sup_{B_0(R)} \left(\hat{u}_\alpha^{2^*} + |\mathcal{L}_\xi \hat{W}_\alpha|_\xi \right) \le 1 + o(1). \tag{3.16}$$

We now let some $R > 0$ and $x \in B_0(R)$. A Green's formula for the first equation in (3.13) shows that:

$$\begin{aligned} \hat{u}_\alpha(x) \ge & \int_{B_x(3R)} \frac{1}{(n-2)\omega_{n-1}} \left(|x-y|^{2-n} - (3R)^{2-n} \right) \frac{\hat{a}_\alpha}{\hat{u}_\alpha^{2^*+1}}(y) dy \\ & - \mu_\alpha^2 \int_{B_x(3R)} \frac{1}{(n-2)\omega_{n-1}} \left(|x-y|^{2-n} - (3R)^{2-n} \right) \hat{h}_\alpha(y) \hat{u}_\alpha(y) dy \end{aligned}$$

so that it is easily seen that there holds, for some positive constant C that does not depend on α nor on R:

$$\int_{B_x(2R)} |x-y|^{2-n} \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi^2 (y) dy \le C.$$

As a consequence, there exists $s_\alpha \in (\frac{3}{2}R, 2R)$ such that, up to a subsequence:

$$\int_{\partial B_0(s_\alpha)} \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi^2 (y) d\sigma(y) \le C R^{n-3}. \tag{3.17}$$

Let now $x \in B_0(R)$. By the properties of φ_α in (3.8) there holds, for some positive C:

$$\left| 2^* \mu_\alpha \xi^{kl} \partial_k (\ln \varphi_\alpha)(\mu_\alpha \cdot) \left(\mathcal{L}_\xi \hat{W}_\alpha \right)_{li} \right| \le C \mu_\alpha^2 |y| \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi$$

so that a Green formula for the operator $\overrightarrow{\triangle}_\xi$ in $B_0(2R)$ (see Section 8) along with (3.16) and (3.17) yield, for some positive C and C':

$$\begin{aligned} \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi (x) & \le C \int_{B_0(s_\alpha)} |x-y|^{1-n} \left| \overrightarrow{\triangle}_\xi \hat{W}_\alpha \right|_\xi dy + C \int_{\partial B_0(s_\alpha)} |x-y|^{1-n} \left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi (y) d\sigma(y) \\ & \le C' R \mu_\alpha + \frac{C'}{R}. \end{aligned}$$

In the end we thus obtain that

$$\left| \mathcal{L}_\xi \hat{W}_\alpha \right|_\xi \to 0 \text{ in } C^0_{loc}(\mathbb{R}^n) \tag{3.18}$$

as $\alpha \to +\infty$. Coming back to (3.15) with (3.18) we obtain that

$$\hat{u}_\alpha(0) = 1 + o(1)$$

and independently that

$$\hat{a}_\alpha \to 0 \text{ in } C^0_{loc}(\mathbb{R}^n) \tag{3.19}$$

as $\alpha \to +\infty$. The Harnack inequality as stated in Section 6 shows then that for any compact set $K \subset \mathbb{R}^n$ there exists a positive constant $C(K)$ such that

$$C(K)^{-1} \le \hat{u}_\alpha \le C(K) \text{ in } K. \tag{3.20}$$

By (3.19), (3.20) and standard elliptic theory one gets that $\hat{u}_\alpha \to U$ in $C^1_{loc}(\mathbb{R}^n)$, where U satisfies $U(0) = 1$ and

$$\triangle_\xi U = f_0(x_0) U^{2^*-1},$$

where $x_0 = \lim x_\alpha$. The classification result in Caffarelli-Gidas-Spruck [8] then shows that

$$U(x) = \left(1 + \frac{f_0(x_0)}{n(n-2)}\right)^{1-\frac{n}{2}}.$$

In particular, since 0 is a strict local maxima of U, for α large enough there exists a sequence $y_\alpha \in M$ of critical points of u_α, with $d_g(x_\alpha, y_\alpha) = o(\mu_\alpha)$ and $\mu_\alpha^{\frac{n-2}{2}} u_\alpha(y_\alpha) \to 1$ as $\alpha \to +\infty$. Using (3.6) this contradicts (3.3) applied at y_α and proves that (3.2) must hold, which concludes the proof of the Proposition. □

Proposition 3.1 provides us with a suitable set of concentration points around which to investigate the behavior of u_α. For the following claims of this section we consider two sequences $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$, where $x_\alpha \in M$ and $16\rho_\alpha < i_g(M)$, such that $\nabla u_\alpha(x_\alpha) = 0$,

$$d_g(x_\alpha, x)^n \left(u_\alpha(x)^{2^*} + |\mathcal{L}_g W_\alpha|_g(x)\right) \le C \quad \text{for } x \in B_{x_\alpha}(8\rho_\alpha), \tag{3.21}$$

and such that

$$u_\alpha(x_\alpha) \ge \frac{1}{C} \tag{3.22}$$

for some positive constant C independent of α. An example of such sequences is given by $x_\alpha = x_{i,\alpha}$ and $\rho_\alpha = \frac{1}{16} d_g\left(x_{i,\alpha}, \{x_{j,\alpha}, j \neq i\}\right)$ where $1 \le i \le N_\alpha$ and $x_{i,\alpha}$ is as 3.1. Two cases can occur and are described by each of the following Propositions:

Proposition 3.2. *Assume that* (2.2) *holds and let* $(u_\alpha, W_\alpha)_\alpha$ *be a sequence of solution of* (2.1) *such that* (3.1) *holds. Assume that for some* $C_1 > 0$ *there holds, up to a subsequence:*

$$\rho_\alpha^n \sup_{B_{x_\alpha}(8\rho_\alpha)} \left(u_\alpha^{2^*} + |\mathcal{L}_g W_\alpha|_g\right) \le C_1. \tag{3.23}$$

Then, for any $x \in B_{x_\alpha}(8\rho_\alpha)$, *there holds:*

$$\rho_\alpha^{\frac{n-2}{2}} u_\alpha(x) \ge \frac{1}{C_2}$$

for some positive C_2.

Proof. Let $(B_{x_\alpha}(\delta), \Phi_\alpha)$, $\delta < i_g(M)$, be a conformal chart around x_α satisfying the same properties as the one constructed in (3.8). Let, for any $x \in B_0(8)$,

$$\begin{aligned} \tilde{u}_\alpha(x) &= \rho_\alpha^{\frac{n-2}{2}} \varphi_\alpha(\rho_\alpha x) u_\alpha \circ \Phi_\alpha^{-1}(\rho_\alpha x), \\ \tilde{W}_\alpha(x) &= \rho_\alpha^{n-1} \varphi_\alpha(\rho_\alpha x)^{-\frac{4}{n-2}} \left(\Phi_\alpha\right)_* W_\alpha(\rho_\alpha x). \end{aligned}$$

Using (3.9) and (3.10) $\tilde{u}_\alpha$ satisfies:

$$\triangle_\xi \tilde{u}_\alpha + \rho_\alpha^2 \tilde{h}_\alpha \tilde{u}_\alpha = \tilde{f}_\alpha \tilde{u}_\alpha^{2^*-1} + \left(\rho_\alpha^{2n} \tilde{b}_\alpha + \left|\rho_\alpha^n \tilde{U}_\alpha + \varphi_\alpha^{2^*}(\rho_\alpha x) \mathcal{L}_\xi \tilde{W}_\alpha(x)\right|_\xi^2\right) \tilde{u}_\alpha^{-2^*-1},$$

where $\tilde{h}_\alpha$, $\tilde{f}_\alpha$, $\tilde{b}_\alpha$, $\tilde{U}_\alpha$, are defined as in (3.14) replacing μ_α by ρ_α. Using (3.23) there holds:

$$\rho_\alpha^{2n} \tilde{b}_\alpha + \left|\rho_\alpha^n \tilde{U}_\alpha + \varphi_\alpha^{2^*}(\rho_\alpha x) \mathcal{L}_\xi \tilde{W}_\alpha(x)\right|_\xi^2 \le C \text{ in } B_0(8)$$

for some $C > 0$, so that using the Harnack inequality as stated in Section 6 and (3.22) there holds:

$$\inf_{B_0(8)} \tilde{u}_\alpha \geq \frac{1}{C}$$

which concludes the proof of the Proposition. □

Proposition 3.3. *Assume that* (2.2) *holds and let* $(u_\alpha, W_\alpha)_\alpha$ *be a sequence of solution of* (2.1) *such that* (3.1) *holds. We assume that the assumptions of Theorem 2.1 hold and that*

$$\rho_\alpha^n \sup_{B_{x_\alpha}(8\rho_\alpha)} \left(u_\alpha^{2^*} + |\mathcal{L}_g W_\alpha|_g \right) \to +\infty \qquad \text{as } \alpha \to +\infty. \tag{3.24}$$

We let

$$\mu_\alpha = u_\alpha(x_\alpha)^{-\frac{2}{n-2}}.$$

Then $\mu_\alpha \to 0$ *and there holds:* $\frac{\rho_\alpha}{\mu_\alpha} \to +\infty$ *and* $\rho_\alpha \to 0$ *as* $\alpha \to +\infty$. *If we let in addition, for any* $x \in M$,

$$B_\alpha(x) = \mu_\alpha^{\frac{n-2}{2}} \left(\mu_\alpha^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)} d_g(x_\alpha, x)^2 \right)^{1-\frac{n}{2}},$$

there holds:

$$\sup_{B_{x_\alpha}(\rho_\alpha)} \left| \frac{u_\alpha}{B_\alpha} - 1 \right| \to 0$$

as $\alpha \to +\infty$.

Proposition 3.3 is the main part of the analysis in this paper and we postpone its proof for now. Section 4 is devoted to the proof of Proposition 3.3.

We now conclude the proof of Theorem 2.1, assuming temporarily Proposition 3.3. We let

$$16 d_\alpha = \min_{1 \leq i,j \leq N_\alpha} d_g\left(x_{i,\alpha}, x_{j,\alpha}\right), \tag{3.25}$$

where the $x_{i,\alpha}$ are given by Proposition 3.1. We assume that $d_\alpha \to 0$ as $\alpha \to +\infty$ and that the $x_{i,\alpha}$ have been reordered so that

$$16 d_\alpha = d_g\left(x_{1,\alpha}, x_{2,\alpha}\right). \tag{3.26}$$

Note that this definition only has a meaning if $N_\alpha \geq 2$ up to a subsequence. If $N_\alpha = 1$ we let $d_\alpha = \frac{1}{16} i_g(M)$. Let $(B_{x_{1,\alpha}}(\delta), \Phi_\alpha)$, with $2\delta < i_g(M)$, be a conformal chart centered at $x_{1,\alpha}$ and satisfying $\left(\Phi_\alpha\right)_* g = \varphi_\alpha^{\frac{4}{n-2}} \xi$, where φ_α is as in (3.8). We let

$$\begin{aligned} \check{u}_\alpha(x) &= d_\alpha^{\frac{n-2}{2}} \varphi_\alpha(d_\alpha x) u_\alpha \circ \Phi_\alpha^{-1}(d_\alpha x), \\ \check{W}_\alpha(x) &= d_\alpha^{n-1} \varphi_\alpha(d_\alpha x)^{-\frac{4}{n-2}} \left(\Phi_\alpha\right)_* W_\alpha(d_\alpha x), \end{aligned} \tag{3.27}$$

so that, using (3.9) and (3.11) the sequence $(\check{u}_\alpha)$ satisfies in $B_0\left(\frac{\delta}{d_\alpha}\right)$:

$$\triangle_\xi \check{u}_\alpha + d_\alpha^2 \check{h}_\alpha \check{u}_\alpha = \check{f}_\alpha \check{u}_\alpha^{2^*-1} + \frac{\check{a}_\alpha}{\check{u}_\alpha^{2^*+1}}, \tag{3.28}$$

where we have let:

$$
\begin{aligned}
&\check{a}_\alpha(x) = d_\alpha^{2n}\check{b}_\alpha(x) + \left| d_\alpha^n \check{U}_\alpha(x) + \varphi_\alpha(d_\alpha x)^{2^*} \mathcal{L}_\xi \check{W}_\alpha \right|_\xi^2 , \\
&\check{h}_\alpha(x) = \varphi_\alpha(d_\alpha x)^{\frac{4}{n-2}} \left(h_\alpha \circ \Phi_\alpha^{-1} - \frac{n-2}{4(n-1)} R(g) \right)(d_\alpha x), \\
&\check{f}_\alpha(x) = f_\alpha \circ \Phi_\alpha^{-1}(d_\alpha x), \\
&\check{b}_\alpha(x) = \varphi_\alpha(d_\alpha x)^{2\cdot 2^*} b_\alpha \circ \Phi_\alpha^{-1}(d_\alpha x), \\
&\check{U}_\alpha(x) = \varphi_\alpha(d_\alpha x)^2 \left(\Phi_\alpha\right)_* U_\alpha(d_\alpha x).
\end{aligned}
\tag{3.29}
$$

For $1 \le i \le N_\alpha$ we let $\check{x}_{i,\alpha} = \frac{1}{d_\alpha}\Phi_\alpha(x_{i,\alpha})$. Let $R > 0$ and define N_R by the following property:

$$
|\check{x}_{i,\alpha}| \le R \iff 1 \le i \le N_R.
$$

By (3.26) there holds $N_R \ge 2$ for R large enough. Propositions 3.2 and 3.3 show that for any $1 \le i \le N_R$ the following alternative occurs:

$$
\begin{aligned}
&\text{either there exists } C_i > 0 \text{ such that } \frac{1}{C_i} \le \check{u}_\alpha \le C_i \text{ in } B_{\check{x}_{i,\alpha}}\left(\frac{1}{2}\right) \\
&\text{or } \sup_{B_{\check{x}_{i,\alpha}}(\frac{1}{2})} \left| \frac{\check{u}_\alpha}{\check{B}_{i,\alpha}} - 1 \right| \to 0 \text{ as } \alpha \to +\infty,
\end{aligned}
\tag{3.30}
$$

where we have let

$$
\check{B}_{i,\alpha}(x) = \check{\mu}_{i,\alpha}^{\frac{n-2}{2}} \left(\check{\mu}_{i,\alpha}^2 + \frac{f_\alpha(x_{i,\alpha})}{n(n-2)} |x - \check{x}_{i,\alpha}|^2 \right)^{1-\frac{n}{2}} \tag{3.31}
$$

and $\check{\mu}_{i,\alpha} = \check{u}_\alpha(\check{x}_{i,\alpha})^{-\frac{2}{n-2}}$. By Proposition 3.3 there holds $\check{\mu}_{i,\alpha} \to 0$ as $\alpha \to +\infty$. At any point $\check{x}_{i,\alpha}$ these alternatives are exclusive and the second one occurs only when $\check{u}_\alpha(\check{x}_{i,\alpha}) \to +\infty$ or when $\check{u}_\alpha \to 0$ in $C^0\left(B_{\check{x}_{i,\alpha}}\left(\frac{1}{2}\right) \backslash B_{\check{x}_{i,\alpha}}\left(\frac{1}{4}\right)\right)$. We start proving that either the first alternative in (3.30) holds for any $1 \le i \le N_R$ or the second alternative holds for all $1 \le i \le N_R$. Let $x \in B_0(R)$. We let $\check{G}_\alpha(x, \cdot)$ be the Green function of the operator $\triangle_\xi + d_\alpha^2 \check{h}_\alpha$ in $B_x(3R)$. Since $\triangle_g + h_0$ is coercive there holds (see Robert [37]) that for any $y \in B_x(2R)$,

$$
\check{G}_\alpha(x, y) \ge \frac{1}{C}
$$

for some positive C. A Green formula thus shows that:

$$
\check{u}_\alpha(x) \ge \frac{1}{C} \int_{B_x(2R)} \check{f}_\alpha(y) \check{u}_\alpha(y)^{2^*-1} dy,
$$

so that if there exists $1 \le i_0 \le N_R$ for which the first alternative in (3.30) is satisfied then there holds:

$$
\check{u}_\alpha(x) \ge C(R) > 0
$$

for some positive $C(R)$ depending on n and on R. Hence the first alternative in (3.30) is satisfied in each $\check{x}_{i,\alpha}$, $1 \le i \le N_R$.

We claim now that the second alternative in (3.30) cannot hold for all $1 \le i \le N_R$. We let $x \in B_0(R)$ and write once again a Green formula as above: there holds

$$\check{u}_\alpha(x) \ge \int_{B_0\left(\frac{1}{2}\right)} \check{G}_\alpha(x,y) \check{f}_\alpha(y) \check{u}_\alpha(y)^{2^*-1} dy + \int_{B_{\check{x}_{2,\alpha}}\left(\frac{1}{2}\right)} \check{G}_\alpha(x,y) \check{f}_\alpha(y) \check{u}_\alpha(y)^{2^*-1} dy.$$

Since $\check{G}_\alpha(x,\cdot)$ converges to the Green function of $\triangle_\xi$ in $B_x(3R)$ in $C^1_{loc}(B_x(3R)\backslash\{x\})$ standard computations yield:

$$\check{u}_\alpha(x) \ge (1+o(1)) \left(\check{B}_{1,\alpha}(x) + \check{B}_{2,\alpha}(x)\right) - \frac{C}{R^{n-2}} \left(\mu_{1,\alpha}^{\frac{n-2}{2}} + \mu_{2,\alpha}^{\frac{n-2}{2}}\right) \tag{3.32}$$

for some positive C that does not depend on R nor on α. We assume now that $|x| \le \frac{1}{4}$ and $x \neq 0$. Applying the second alternative in (3.30) at 0 yields, with (3.32) and (3.31):

$$\left(\frac{\mu_{1,\alpha}}{\mu_{2,\alpha}}\right)^{\frac{n-2}{2}} |x|^{n-2} \Big(|x - \check{x}_2|^{2-n} - CR^{2-n} + o(1) \Big) \le o(1) + \frac{C}{R^{n-2}} |x|^{2-n},$$

where $\check{x}_2 = \lim \check{x}_{2,\alpha}$, so that $|\check{x}_2| = 16$ by (3.26). In the end letting $\alpha \to +\infty$, dividing by $|x|$ and letting $x \to 0$ yields, for R large enough:

$$\limsup_{\alpha \to +\infty} \left(\frac{\mu_{1,\alpha}}{\mu_{2,\alpha}}\right)^{\frac{n-2}{2}} \le C \frac{16^{n-2}}{R^{n-2} - C16^{n-2}}. \tag{3.33}$$

The same arguments as those developed to obtain (3.33) work in the same way exchanging the roles of $\check{x}_{1,\alpha}$ and $\check{x}_{2,\alpha}$ so that in the end we also obtain:

$$\limsup_{\alpha \to +\infty} \left(\frac{\mu_{2,\alpha}}{\mu_{1,\alpha}}\right)^{\frac{n-2}{2}} \le C \frac{16^{n-2}}{R^{n-2} - C16^{n-2}}.$$

This clearly contradicts (3.33) up to choosing R large enough. So far we have proven that for R large enough and for any $1 \le i \le N_R$ the first alternative in (3.30) holds at every $\check{x}_{i,\alpha}$. Note in particular that by Proposition 3.2 there also holds $\left|\mathcal{L}_\xi \check{W}_\alpha\right|_\xi \le C_i$ in $B_{\check{x}_{i,\alpha}}\left(\frac{1}{2}\right)$. Thus, by Proposition 3.1 and by the first alternative in (3.30) there holds that $\check{u}_\alpha$ and $\left|\mathcal{L}_\xi \check{W}_\alpha\right|_\xi$ belong to $L^\infty_{loc}(\mathbb{R}^n)$. Mimicking the arguments in the proof of Proposition 3.1 that led to (3.18) we have here again:

$$\left|\mathcal{L}_\xi \check{W}_\alpha\right|_\xi \to 0 \text{ in } C^0_{loc}(\mathbb{R}^n)$$

so that, as in the proof of Proposition 3.1, we obtain that $\check{u}_\alpha \to \check{U}$ in $C^1_{loc}(\mathbb{R}^n)$, where U satisfies

$$\triangle_\xi U = f_0(x_1) U^{2^*-1},$$

with $x_1 = \lim x_{1,\alpha}$ and where $U(x_1) > 0$ since the first alternative of (3.30) holds at 0. The classification result of Caffarelli-Gidas-Spruck [8] thus shows that U has only a critical point, but this is impossible since $\check{u}_\alpha$ had at least two critical points, namely 0 and $\check{x}_{2,\alpha}$, and $|\check{x}_{2,\alpha}| = 16$. This therefore contradicts the fact that $d_\alpha \to 0$ as $\alpha \to +\infty$.

Thus there exists some $\delta_0 > 0$ such that $d_\alpha \ge \delta_0$. In particular the $x_{i,\alpha}$ constructed in Proposition 3.1 are in finite number and around each of them u_α is

bounded, since the situation of Proposition 3.3 cannot happen because $d_\alpha \not\to 0$. In the end, u_α is bounded in M and Theorem 2.1 is proven.

4. Blow-up analysis – Proof of Proposition 3.3

In this section we develop the asymptotic blow-up analysis needed to prove Proposition 3.3. We let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of (2.1) and assume that (2.2) and the assumptions of Theorem 2.1 hold. We let $(x_\alpha)_\alpha$ be a sequence of critical points of u_α in M and $(\rho_\alpha)_\alpha$ be a sequence of positive real numbers with $16\rho_\alpha < i_g(M)$ such that

$$d_g(x_\alpha, x)^n \left(u_\alpha(x)^{2^*} + |\mathcal{L}_g W_\alpha|_g(x)\right) \le C \quad \text{for } x \in B_{x_\alpha}(8\rho_\alpha), \tag{4.1}$$

and such that

$$\rho_\alpha^n \sup_{B_{x_\alpha}(8\rho_\alpha)} \left(u_\alpha^{2^*} + |\mathcal{L}_g W_\alpha|_g\right) \to +\infty \quad \text{as } \alpha \to +\infty. \tag{4.2}$$

In this section, the letter C will always denote a positive constant, whose value may change from one line to another but that will never depend on α. Let $(B_{x_\alpha}(16\rho_\alpha), \Phi_\alpha)$ be some conformal chart around x_α as in (3.8), such that in the chart there holds $g_{ij} = \varphi_\alpha^{\frac{4}{n-2}} \xi_{ij}$ for some conformal factor φ_α . We can always assume that $B_0(8\rho_\alpha) \subset \Phi_\alpha(B_{x_\alpha}(16\rho_\alpha))$ and that the sequence of conformal factors $(\varphi_\alpha)_\alpha$ is uniformly bounded in $C^k(B_0(8\rho_\alpha))$ for any $k \ge 0$. We let, for any $x \in B_0(8\rho_\alpha)$:

$$\begin{aligned} v_\alpha(x) &= \varphi_\alpha(x) u_\alpha \circ \Phi_\alpha^{-1}(x), \\ Z_\alpha(x) &= \varphi_\alpha(x)^{-\frac{4}{n-2}} \left(\Phi_\alpha\right)_* W_\alpha(x). \end{aligned} \tag{4.3}$$

Using (3.9) and (3.11) the sequence $(v_\alpha, Z_\alpha)_\alpha$ satisfies the following system of equations in $B_0(8\rho_\alpha)$:

$$\begin{cases} \triangle_\xi v_\alpha + \tilde{h}_\alpha v_\alpha = \tilde{f}_\alpha v_\alpha^{2^*-1} + \dfrac{\tilde{a}_\alpha}{v_\alpha^{2^*+1}}, \\ \left(\overrightarrow{\triangle}_\xi Z_\alpha\right)_i = 2^* \xi^{kl} \partial_k (\ln \varphi_\alpha) \left(\mathcal{L}_\xi Z_\alpha\right)_{li} + v_\alpha^{2^*} \left(\tilde{X}_\alpha\right)_i + \left(\tilde{Y}_\alpha\right)_i, \end{cases} \tag{4.4}$$

where we have let:

$$\begin{aligned} \tilde{a}_\alpha(x) &= \tilde{b}_\alpha(x) + \left|\tilde{U}_\alpha(x) + \varphi_\alpha(x)^{2^*} \mathcal{L}_\xi Z_\alpha\right|_\xi^2, \\ \tilde{h}_\alpha(x) &= \varphi_\alpha(x)^{\frac{4}{n-2}} \left(h_\alpha \circ \Phi_\alpha^{-1} - \frac{n-2}{4(n-1)} R(g)\right)(x), \\ \tilde{f}_\alpha(x) &= f_\alpha \circ \Phi_\alpha^{-1}(x), \\ \tilde{b}_\alpha(x) &= \varphi_\alpha(x)^{2\cdot 2^*} b_\alpha \circ \Phi_\alpha^{-1}(x), \\ \tilde{U}_\alpha(x) &= \varphi_\alpha(x)^2 \left(\Phi_\alpha\right)_* U_\alpha(x), \\ \tilde{X}_\alpha(x) &= \varphi_\alpha(x)^{-2^*} \left(\Phi_\alpha\right)_* X_\alpha(x) \text{ and} \\ \tilde{Y}_\alpha(x) &= \left(\Phi_\alpha\right)_* Y_\alpha(x). \end{aligned} \tag{4.5}$$

As before, note that by (3.8) there holds, for $x \in B_0(8\rho_\alpha)$:

$$\left|2^* \xi^{kl} \partial_k (\ln \varphi_\alpha) \left(\mathcal{L}_\xi Z_\alpha\right)_{li}\right| \le C|y| \left|\mathcal{L}_\xi Z_\alpha\right|_\xi , \tag{4.6}$$

and (4.1) becomes: for all $x \in B_0(4\rho_\alpha)$,

$$|x|^n \left(v_\alpha^{2^*}(x) + |\mathcal{L}_\xi Z_\alpha|_\xi(x) \right) \le C. \tag{4.7}$$

4.1. **Sharp pointwise estimates.** We first prove a local version of Proposition 3.3.

Claim 4.1. *Let*

$$\mu_\alpha = u_\alpha(x_\alpha)^{-\frac{2}{n-2}} = v_\alpha(0)^{-\frac{2}{n-2}}. \tag{4.8}$$

There holds, up to a subsequence: $\mu_\alpha \to 0$, $\frac{\rho_\alpha}{\mu_\alpha} \to +\infty$,

$$\mu_\alpha^{\frac{n-2}{2}} v_\alpha(\mu_\alpha x) \to U(x) = \left(1 + \frac{f_0(x_0)}{n(n-2)}|x|^2\right)^{1-\frac{n}{2}} \quad \text{in } C^{1,\eta}_{loc}(\mathbb{R}^n),$$

and

$$\mu_\alpha^n \left|\mathcal{L}_\xi Z_\alpha\right|_\xi (\mu_\alpha x) \to 0 \text{ in } C^0_{loc}(\mathbb{R}^n)$$

as $\alpha \to +\infty$, *where* $x_0 = \lim_{\alpha\to\infty} x_\alpha$.

Proof. The proof of Claim 4.1 is similar to that of Proposition 3.1. Let $y_\alpha \in B_{x_\alpha}(8\rho_\alpha)$ be a sequence of points of M satisfying

$$u_\alpha^{2^*}(y_\alpha) + \left|\mathcal{L}_g W_\alpha\right|_g (y_\alpha) = \sup_{B_{x_\alpha}(8\rho_\alpha)} \left(u_\alpha^{2^*} + \left|\mathcal{L}_g W_\alpha\right|_g \right), \tag{4.9}$$

and let

$$\nu_\alpha^{-n} = u_\alpha^{2^*}(y_\alpha) + \left|\mathcal{L}_g W_\alpha\right|_g (y_\alpha). \tag{4.10}$$

By (4.2) there holds

$$\frac{\rho_\alpha}{\nu_\alpha} \to +\infty \tag{4.11}$$

as $\alpha \to \infty$, so that in particular there holds

$$\nu_\alpha \to 0 \tag{4.12}$$

as $\alpha \to +\infty$. By (4.1) we also get:

$$d_g(y_\alpha, x_\alpha) \le C\nu_\alpha \tag{4.13}$$

for some positive C. We let, for any $x \in B_0\left(\frac{8\rho_\alpha}{\nu_\alpha}\right)$:

$$\begin{aligned} \hat{v}_\alpha(x) &= \nu_\alpha^{\frac{n-2}{2}} v_\alpha(\nu_\alpha x), \\ \hat{Z}_\alpha(x) &= \nu_\alpha^{n-1} Z_\alpha(\nu_\alpha x), \end{aligned} \tag{4.14}$$

where v_α and Z_α are as in (4.3). It is easily seen that $\hat{v}_\alpha$ and $\hat{Z}_\alpha$ satisfy:

$$\begin{cases} \triangle_\xi \hat{v}_\alpha + \nu_\alpha^2 \hat{h}_\alpha \hat{v}_\alpha = \hat{f}_\alpha \hat{v}_\alpha^{2^*-1} + \dfrac{\hat{a}_\alpha}{\hat{v}_\alpha^{2^*+1}}, \\ \left(\overrightarrow{\triangle}_\xi \hat{Z}_\alpha\right)_i = 2^* \nu_\alpha \xi^{kl} \partial_k (\ln \varphi_\alpha)(\nu_\alpha x) \left(\mathcal{L}_\xi \hat{Z}_\alpha\right)_{li} + \nu_\alpha \hat{v}_\alpha^{2^*} \left(\hat{X}_\alpha\right)_i + \nu_\alpha^{n+1} \left(\tilde{Y}_\alpha\right)_i, \end{cases}$$

where we have let:

$$\begin{aligned}
&\hat{a}_\alpha(x) = \nu_\alpha^{2n}\hat{b}_\alpha(x) + \left|\nu_\alpha^n \hat{U}_\alpha(x) + \varphi_\alpha(\nu_\alpha x)^{2^*}\mathcal{L}_\xi \hat{Z}_\alpha\right|_\xi^2, \\
&\hat{h}_\alpha(x) = \tilde{h}_\alpha(\nu_\alpha x), \\
&\hat{f}_\alpha(x) = \tilde{f}_\alpha(\nu_\alpha x), \\
&\hat{b}_\alpha(x) = \tilde{b}_\alpha(\nu_\alpha x), \\
&\hat{U}_\alpha(x) = \tilde{U}_\alpha(\nu_\alpha x), \\
&\hat{X}_\alpha(x) = \tilde{X}_\alpha(\nu_\alpha x), \\
&\hat{Y}_\alpha(x) = \tilde{Y}_\alpha(\nu_\alpha x).
\end{aligned} \tag{4.15}$$

By definition of $\hat{v}_\alpha$ and $\hat{Z}_\alpha$ there holds: for any $x \in B_0\left(\frac{8\rho_\alpha}{\mu_\alpha}\right)$

$$\hat{v}_\alpha^{2^*}(x) + \left|\mathcal{L}_\xi \hat{Z}_\alpha\right|_\xi (x) \le 1 = \hat{v}_\alpha^{2^*}(\hat{y}_\alpha) + \left|\mathcal{L}_\xi \hat{Z}_\alpha\right|_\xi (\hat{y}_\alpha), \tag{4.16}$$

where we have let $\hat{y}_\alpha = \frac{1}{\nu_\alpha}\Phi_\alpha(y_\alpha)$. Let $R > 0$ and $x \in B_0(R)$. Mimicking the arguments used in the proof of Proposition 3.1 to obtain (3.17), it is easily seen that there exists a positive constant C that does not depend on α nor on R and a sequence $s_\alpha \in (\frac{3}{2}R, 2R)$ such that

$$\int_{\partial B_0(s_\alpha)} \left|\mathcal{L}_\xi \hat{Z}_\alpha\right|_\xi^2 (y) d\sigma(y) \le C R^{n-3}. \tag{4.17}$$

As a consequence, using (4.17), the arguments in the proof of Proposition 3.1 that led to (3.18) still apply here and show that there holds:

$$\left|\mathcal{L}_\xi \hat{Z}_\alpha\right|_\xi \to 0 \text{ in } C^0_{loc}(\mathbb{R}^n) \tag{4.18}$$

as $\alpha \to +\infty$. By (4.18) we therefore obtain with (4.16) that $\hat{v}_\alpha(\hat{y}_\alpha) = 1 + o(1)$ and that

$$\hat{a}_\alpha(x) \to \text{ in } C^0_{loc}(\mathbb{R}^n),$$

where $\hat{a}_\alpha(x)$ is as in (4.15). By (4.13), there exists $\hat{y}_0 \in \mathbb{R}^n$ such that $\hat{y}_\alpha \to \hat{y}_0$ as $\alpha \to +\infty$. Here again, the Harnack inequality of Section 6 shows that

$$\hat{v}_\alpha \to U \text{ in } C^1_{loc}(\mathbb{R}^n), \tag{4.19}$$

where U satisfies $\nabla U(0) = 0$ by definition of x_α, $U(\hat{y}_0) = 1$ and:

$$\triangle_\xi U = f_0(x_0) U^{2^*-1}.$$

The classification result in Caffarelli-Gidas-Spruck therefore shows that $U(x) = \left(1 + \frac{f_0(x_0)}{n(n-2)}|x|^2\right)^{1-\frac{n}{2}}$. In particular, $\hat{y}_0 = 0$, and since there independently holds by (4.8) that

$$\hat{v}_\alpha(\hat{y}_\alpha) = \left(\frac{\nu_\alpha}{\mu_\alpha}\right)^{\frac{n-2}{2}},$$

we obtain in the end

$$\lim_{\alpha\to\infty} \frac{\nu_\alpha}{\mu_\alpha} = 1,$$

which, with (4.19), (4.11) and (4.12), concludes the proof of Claim 4.1. $\square$

We set in the following, for $x \in \mathbb{R}^n$:

$$B_\alpha(x) = \mu_\alpha^{\frac{n-2}{2}} \left(\mu_\alpha^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)} |x|^2 \right)^{1-\frac{n}{2}}, \tag{4.20}$$

where μ_α is as in (4.8). It satisfies $\triangle_\xi B_\alpha = f_\alpha(x_\alpha) B_\alpha^{2^*-1}$ in $\mathbb{R}^n$.

Claim 4.1 establishes Proposition 3.3 when the distance to the concentration point 0 is of order μ_α. In order to prove Proposition 3.3 we have to extend the estimate on v_α given by Claim 4.1 to the whole ball $B_0(\rho_\alpha)$. We need to show that the bubble B_α is dominant with respect to other possible defects of compactness of the sequence $(v_\alpha)_\alpha$ up to a radius ρ_α. The standard way to proceed would consist in showing that ρ_α coincides with the maximal radius up to which 0 is an isolated simple blow-up point of v_α, following standard terminology. The usual arguments, however, fail to work here. One reason is that the estimates we have on v_α and $\mathcal{L}_\xi Z_\alpha$ at this point – given by Claim 4.1 and the weak estimate (4.7) – do not improve because of the highly non-trivial coupling of system (2.1).

We therefore proceed in a different way. Let $\varepsilon > 0$ be given. We define the radius of influence r_α of the bubble centered at 0 to be the maximal radius where v_α looks almost like a bubble B_α, up to an ε error factor: precisely

$$r_\alpha = \sup \mathcal{R} \tag{4.21}$$

where

$$\mathcal{R} = \Bigg\{ 0 < r \le \rho_\alpha \text{ such that } v_\alpha \le (1+\varepsilon) B_\alpha, \quad |\nabla (v_\alpha - B_\alpha)|_\xi \le \varepsilon |\nabla B_\alpha|_\xi \\ \text{and } x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha \le 0 \text{ on } B_0(r) \backslash B_0(2 R_\alpha \mu_\alpha) \Bigg\}, \tag{4.22}$$

where we have let

$$R_\alpha^2 = \frac{n(n-2)}{f_\alpha(x_\alpha)}. \tag{4.23}$$

Using Claim 4.1 there easily holds

$$\frac{r_\alpha}{\mu_\alpha} \to +\infty \tag{4.24}$$

as $\alpha \to +\infty$. Because of the definition of r_α, in $B_0(r_\alpha)$ the unknown v_α looks almost like what we believe to be its sharp asymptotic. We will prove Proposition 3.24 by proving that $r_\alpha = \rho_\alpha$. To do this we have to obtain sharp pointwise asymptotic estimates on v_α and $\mathcal{L}_\xi Z_\alpha$ in all of $B_0(r_\alpha)$. To reach the desired precision we improve the estimates on v_α and $\mathcal{L}_\xi Z_\alpha$ step by step, by performing some kind of ping-pong game: we plug the available estimates on the unknowns in system (2.1) and, through several Green representation formulae, iteratively recover better estimates for both the unknowns.

We start improving the information on v_α in $B_0(8 r_\alpha)$:

Claim 4.2. *Let $(\delta_\alpha)_\alpha$, $0 < \delta_\alpha \le r_\alpha$ be some sequence of radii. There exists a sequence $(\kappa_\alpha)_\alpha$, $0 \le \kappa_\alpha < 1$, of real numbers such that for any $z_\alpha \in B_0(8\delta_\alpha)$ there holds:*

$$(1 - \kappa_\alpha) B_\alpha(z_\alpha) \le v_\alpha(z_\alpha) \le C B_\alpha(z_\alpha), \tag{4.25}$$

where $C > 1$ is a constant that does not depend on α and B_α is as in (4.20). Furthermore, if $\delta_\alpha \to 0$ as $\alpha \to \infty$, there holds $\kappa_\alpha \to 0$.

Proof. We start proving the upper bound. We define, for $x \in B_0(8)$:

$$\bar{v}_\alpha(x) = r_\alpha^{\frac{n-2}{2}} v_\alpha(r_\alpha x),$$

where v_α and r_α are defined in (4.3) and (4.21). Then $\bar{v}_\alpha$ satisfies:

$$\triangle_\xi \bar{v}_\alpha + r_\alpha^2 \bar{h}_\alpha(x)\bar{v}_\alpha = \bar{f}_\alpha \bar{v}_\alpha^{2^*-1} + r_\alpha^{2n} \frac{\bar{a}_\alpha}{\bar{v}_\alpha^{2^*+1}},$$

where, using the notations in (4.5):

$$\begin{aligned} \bar{a}_\alpha(x) &= \tilde{a}_\alpha(r_\alpha x), \\ \bar{h}_\alpha(x) &= \tilde{h}_\alpha(r_\alpha x), \\ \bar{f}_\alpha(x) &= \tilde{f}_\alpha(r_\alpha x). \end{aligned}$$

By the definition of r_α there holds, for some positive C that does not depend on α:

$$\bar{v}_\alpha(x) \le C \left(\frac{\mu_\alpha}{r_\alpha}\right)^{\frac{n-2}{2}} \text{ in } B_0(1)\backslash B_0\left(\frac{1}{2}\right).$$

The weak estimate (4.7) shows both that:

$$\bar{v}_\alpha \le C \quad \text{and} \quad r_\alpha^{2n}\bar{a}_\alpha \le C \qquad \text{in } B_0(8)\backslash B_0\left(\frac{1}{2}\right).$$

Hence the Harnack inequality, as stated in Section 6, successively applied to the annuli $B_0(3/2)\backslash B_0(1/2), \dots, B_0(17/2)\backslash B_0(11/2)$ shows that:

$$\bar{v}_\alpha(x) \le C \left(\frac{\mu_\alpha}{r_\alpha}\right)^{\frac{n-2}{2}} \text{ in } B_0(8)\backslash B_0\left(\frac{1}{2}\right),$$

for some positive C. By the definition of r_α in (4.21) and because of (4.24) and the expression of B_α in (4.20) this yields the upper bound in (4.25).

For the lower bound, we let G_α be the Green function of $\triangle_g + h_\alpha$ in M. Using the expression of v_α in (4.3) and the notations of (4.5) we have that, for any sequence z_α of points in $B_0(8r_\alpha)$:

$$v_\alpha(z_\alpha) \ge \varphi_\alpha(z_\alpha) \int_{B_0(8r_\alpha)} \varphi_\alpha(y) G_\alpha\left(\Phi_\alpha^{-1}(z_\alpha), \Phi_\alpha^{-1}(y)\right) \tilde{f}_\alpha(y) v_\alpha^{2^*-1}(y) dy.$$

In particular, with the expression of B_α as in (4.20) we can write that:

$$\begin{aligned} \frac{v_\alpha}{B_\alpha}(z_\alpha) \ge \varphi_\alpha(z_\alpha) \int_{B_0\left(\frac{6r_\alpha}{\mu_\alpha}\right)} & \varphi_\alpha(\mu_\alpha y)\tilde{f}_\alpha(\mu_\alpha y)\left(\mu_\alpha^{\frac{n-2}{2}} v_\alpha(\mu_\alpha y)\right)^{2^*-1} \\ & \times G_\alpha\left(\Phi_\alpha^{-1}(z_\alpha), \Phi_\alpha^{-1}(\mu_\alpha y)\right) d_g\left(\Phi_\alpha^{-1}(z_\alpha), \Phi_\alpha^{-1}(\mu_\alpha y)\right)^{n-2} \\ & \times \left(\frac{\mu_\alpha^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)}|z_\alpha|^2}{d_g\left(\Phi_\alpha^{-1}(z_\alpha), \Phi_\alpha^{-1}(\mu_\alpha y)\right)^2}\right)^{\frac{n-2}{2}} dy, \end{aligned}$$

which yields the lower bound in (4.25) using Claim 4.1, Fatou's lemma and standard properties of Green functions (see Robert [37]). $\square$

Exploiting the coupling of the system allows one to recover an integral control on $\mathcal{L}_\xi Z_\alpha$, better than the one given by the weak estimate (4.7). This issue is addressed in the next Claim:

Claim 4.3. *Let $(\delta_\alpha)_\alpha$ be a sequence of positive numbers satisfying*

$$\frac{\mu_\alpha}{\delta_\alpha} \to 0 \quad \text{and} \quad \delta_\alpha \le \min(r_\alpha, \mu_\alpha^{\frac{1}{2}}). \tag{4.26}$$

There holds then, for some positive constant C, that for any $x \in B_0(7\delta_\alpha)$,

$$\int_{B_0(6\delta_\alpha)} |x-y|^{2-n} \left|\mathcal{L}_\xi Z_\alpha\right|_\xi^2 v_\alpha^{-2^*-1}(y)dy \le C B_\alpha(x). \tag{4.27}$$

As a consequence:

$$\int_{B_0(6\delta_\alpha)\backslash B_0(\delta_\alpha)} \left|\mathcal{L}_\xi Z_\alpha\right|_\xi^2 dy \le C\mu_\alpha^{2n-2}\delta_\alpha^{2-3n}, \tag{4.28}$$

and there exists a sequence of positive numbers $s_\alpha \in (5\delta_\alpha, 6\delta_\alpha)$ such that

$$\int_{\partial B(s_\alpha)} \left|\mathcal{L}_\xi Z_\alpha\right|_\xi (y)^2 d\sigma(y) \le C\mu_\alpha^{2n-2}\delta_\alpha^{1-3n}. \tag{4.29}$$

Proof. Using (4.4) we write a Green formula for $\triangle_\xi + \tilde{h}_\alpha$ in $B_0(8\delta_\alpha)$ to obtain that for some $C > 0$ there holds, for any $x \in B_0(7\delta_\alpha)$:

$$v_\alpha(x) \ge \frac{1}{C}\int_{B_0(6\delta_\alpha)} |x-y|^{2-n}\left(\tilde{b}_\alpha + \left|\tilde{U}_\alpha + \varphi^{2^*}\mathcal{L}_\xi Z_\alpha\right|_\xi^2\right) v_\alpha^{-2^*-1}(y)dy.$$

Using (4.25) and (4.26) yields (4.27) since $\tilde{b}_\alpha \ge 0$. As a consequence, choosing $|x| = 7\delta_\alpha$ in (4.27) there holds

$$\int_{B_0(6\delta_\alpha)\backslash B_0(\lambda\delta_\alpha)} \left|\mathcal{L}_\xi Z_\alpha\right|_\xi^2 v_\alpha^{-2^*-1}(y)dy \le C\mu_\alpha^{\frac{n-2}{2}}$$

for any $1 \le \lambda \le 6$ and (4.28) and (4.29) follow then using once again (4.25). $\square$

The goal of our analysis is to obtain sharp pointwise estimates on v_α and $|\mathcal{L}_\xi Z_\alpha|_\xi$. The two unknowns may blow-up simultaneously but at different rates and we are then led to the simultaneous investigation of their defects of compactness. Let us define now the following 1-forms in $\mathbb{R}^n$ by:

$$\begin{aligned} V_\alpha(x)_i &= \tilde{X}_\alpha(0)^j \int_{\mathbb{R}^n} B_\alpha^{2^*}(y)\mathcal{G}_i(x-y)_j dy, \\ P_{\alpha,k}(x)_i &= \partial_k\tilde{X}_\alpha(0)^j \int_{\mathbb{R}^n} y_k B_\alpha^{2^*}(y)\mathcal{G}_i(x-y)_j dy \quad \text{ for } 1 \le k \le n, \end{aligned} \tag{4.30}$$

where, for $y \ne 0$,

$$\mathcal{G}_i(y)_j = -\frac{1}{4(n-1)\omega_{n-1}}|y|^{2-n}\left((3n-2)\delta_{ij} + (n-2)\frac{y_i y_j}{|y|^2}\right) \tag{4.31}$$

is the i-th fundamental solution of $\overrightarrow{\triangle}_\xi$ in $\mathbb{R}^n$, see Section 8. Then V_α and $P_{\alpha,k}$ satisfy in $\mathbb{R}^n$:

$$\overrightarrow{\triangle}_\xi V_\alpha = v_\alpha^{2^*}\tilde{X}_\alpha(0) \quad \text{and} \quad \overrightarrow{\triangle}_\xi P_{\alpha,k} = y_k B_\alpha^{2^*}\partial_k\tilde{X}_\alpha(0). \tag{4.32}$$

We also let in the following:

$$\varepsilon_\alpha = |\tilde{X}_\alpha(0)|_\xi, \quad \beta_{\alpha,k} = |\partial_k \tilde{X}_\alpha(0)|_\xi, \quad \beta_\alpha = \sum_{k=1}^n \beta_{\alpha,k} \tag{4.33}$$

and

$$\zeta_0 = \lim_{\alpha\to\infty} \frac{\tilde{X}_\alpha(0)}{\varepsilon_\alpha}, \quad \zeta_k = \lim_{\alpha\to\infty} \frac{\partial_k \tilde{X}_\alpha(0)}{\beta_{\alpha,k}} \text{ for } 1 \le k \le n, \tag{4.34}$$

which are all vectors of Euclidean norm 1. To define the ζ_k we should require ε_α and $\beta_{\alpha,k}$ to be nonzero. This will however not be an issue since the vectors ζ_k always appear in the computations multiplied by ε_α or $\beta_{\alpha,k}$. Define, for any $z_\alpha \in B_0(8r_\alpha)$:

$$\theta_\alpha(z_\alpha) = \left(\mu_\alpha^2 + |z_\alpha|^2\right)^{\frac{1}{2}}. \tag{4.35}$$

It is easily seen that there holds, for any $x \in \mathbb{R}^n$:

$$|\mathcal{L}_\xi V_\alpha|_\xi \le C\varepsilon_\alpha \theta_\alpha(x)^{1-n}, \tag{4.36}$$

and

$$|\mathcal{L}_\xi P_{\alpha,k}|_\xi \le C\beta_{\alpha,k}\mu_\alpha^2 \theta_\alpha(x)^{-n} \tag{4.37}$$

where $\theta_\alpha(x)$ is as in (4.35). Moreover, if $\varepsilon_\alpha \ne 0$ and $\beta_\alpha \ne 0$ straightforward computations from (4.30) give the following asymptotic expansion of V_α and $P_{\alpha,k}$: for any sequence $z_\alpha \in B_0(8\rho_\alpha)$ satisfying $\frac{|z_\alpha|}{\mu_\alpha} \to +\infty$ there holds, up to a subsequence:

$$\begin{aligned} \mathcal{L}_\xi V_\alpha(z_\alpha)_{ij} = C_1(n) f_\alpha(x_\alpha)^{-\frac{n}{2}} & \left[\delta_{ij}\zeta^p \check{z}_p - \zeta_i \check{z}_j - \zeta_j \check{z}_i - (n-2)\zeta^p \check{z}_p \check{z}_i \check{z}_j \right] \\ & \times \varepsilon_\alpha |z_\alpha|^{1-n}(1+o(1)), \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} \mathcal{L}_\xi P_{\alpha,k}(z_\alpha)_{ij} &= C_2(n) f_\alpha(x_\alpha)^{-\frac{n+2}{2}} \Bigg\{ -(\zeta_k)_i \left[n\check{z}_j\check{z}_k - \delta_{ij}\right] \\ &+ \left[n\delta_{ij}\check{z}_k - (n-2)\delta_{ik}\check{z}_j - (n-2)\delta_{jk}\check{z}_i + (n+2)(n-2)\check{z}_i\check{z}_j\check{z}_k \right] \langle \zeta_k, \check{z}\rangle_\xi \\ &+ (\zeta_k)_j \left[n\check{z}_i\check{z}_k - \delta_{ik}\right] - (\zeta_k)_k \left[(n-2)\check{z}_i\check{z}_j + \delta_{ij} \right] \Bigg\} \\ &\times \beta_{\alpha,k}\mu_\alpha^2 |z_\alpha|^{-n}\big(1+o(1)\big) \end{aligned} \tag{4.39}$$

where $\check{z} = \lim_{\alpha\to+\infty} \frac{z_\alpha}{|z_\alpha|}$,

$$C_1(n) = \frac{n^{\frac{n+2}{2}}(n-2)^{\frac{n}{2}}\omega_n}{2^{n+1}(n-1)\omega_{n-1}}, \quad C_2(n) = -\frac{n^{\frac{n+2}{4}}(n-2)^{\frac{n}{2}}\omega_n}{2^{n+1}(n-1)\omega_{n-1}},$$

and where ω_d stands for the area of the d-dimensional sphere in $\mathbb{R}^{d+1}$.

We show in what follows that the 1-forms V_α and $P_{\alpha,k}$ respectively appear as the first and second-order terms in the asymptotic development of Z_α. We start estimating the difference $|\mathcal{L}_\xi(Z_\alpha - V_\alpha)|_\xi$.

Claim 4.4. *Let $(\delta_\alpha)_\alpha$ be a sequence of positive numbers satisfying:*

$$\frac{\mu_\alpha}{\delta_\alpha} \to 0 \quad \text{and} \quad \delta_\alpha \le \min(r_\alpha, \mu_\alpha^{\frac{1}{2}}). \tag{4.40}$$

There holds, for any sequence $z_\alpha \in B_0(3\delta_\alpha)$:

$$\begin{aligned} \left|\mathcal{L}_\xi \left(Z_\alpha - V_\alpha\right)\right|_\xi (z_\alpha) \le C &\left(\varepsilon_\alpha \delta_\alpha^{1-n} + \mu_\alpha^{n-1}\delta_\alpha^{1-2n} + \eta\left(\frac{\mu_\alpha}{\theta_\alpha(z_\alpha)}\right)\mu_\alpha \theta_\alpha(z_\alpha)^{1-n}\right) \\ &+ o\left(\varepsilon \theta_\alpha(z_\alpha)^{1-n}\right), \end{aligned} \tag{4.41}$$

where $\theta_\alpha(z_\alpha)$ is as in (4.35)*, ε_α is as in* (4.33) *and η is a nonnegative, continuous function in $\mathbb{R}$ with $\eta(0) = 0$. As a consequence, there holds*

$$\varepsilon_\alpha = O\left(\mu_\alpha^{n-1}\delta_\alpha^{-n}\right) + o(\mu_\alpha). \tag{4.42}$$

Proof. Let $(\delta_\alpha)_\alpha$ be a sequence of positive numbers satisfying (4.40). We estimate the difference $\left|\mathcal{L}_\xi \left(Z_\alpha - V_\alpha\right)\right|_\xi (z_\alpha)$ using a Green representation formula. We let $\mathcal{G}_{\alpha,i}$ be the i-th Green 1-form for $\overrightarrow{\triangle}_\xi$ with Neumann boundary condition on $B_0(s_\alpha)$, where s_α is as in Claim 4.3 (see Section 8). A Green representation formula for $\overrightarrow{\triangle}_\xi$ on $B_0(s_\alpha)$ writes then: for any sequence $z_\alpha \in B_0(4\delta_\alpha)$,

$$\begin{aligned} \mathcal{L}_\xi \left(Z_\alpha - V_\alpha\right)_{ij}(z_\alpha) = &\int_{B_0(s_\alpha)} \mathcal{H}_{ij,\alpha}(z_\alpha, y)_p \overrightarrow{\triangle}_\xi \left(Z_\alpha - V_\alpha\right)^p(y) dy \\ &+ \int_{\partial B_0(s_\alpha)} \mathcal{H}_{ij,\alpha}(z_\alpha, y)_p \nu_q \mathcal{L}_\xi \left(Z_\alpha - V_\alpha\right)^{pq}(y) d\sigma(y) \end{aligned} \tag{4.43}$$

where we have let, for $x, y \in B_0(s_\alpha)$:

$$\mathcal{H}_{ij,\alpha}(x,y)_p = \partial_i \mathcal{G}_{\alpha,j}(x,y)_p + \partial_j \mathcal{G}_{\alpha,i}(x,y)_p - \frac{2}{n}\xi_{ij}\sum_{k=1}^n \partial_k \mathcal{G}_{\alpha,k}(x,y)_p \tag{4.44}$$

and the derivatives are taken with respect to x. Since $|z_\alpha| \le 4\delta_\alpha$ and $s_\alpha \ge 5\delta_\alpha$ we can thus write, using (4.4), (4.6), (4.32), (4.36), Claim 4.3 and Claim 8.1 below, that:

$$\left|\mathcal{L}_\xi \left(Z_\alpha - V_\alpha\right)\right|_\xi (z_\alpha) \le C\left(I_1 + I_2 + I_3 + I_4 + \varepsilon_\alpha \delta_\alpha^{1-n} + \mu_\alpha^{n-1}\delta_\alpha^{1-2n}\right) \tag{4.45}$$

where we have let:

$$\begin{aligned} I_1 &= \int_{B_0(6\delta_\alpha)} |z_\alpha - y|^{1-n}|y| \left|\mathcal{L}_\xi Z_\alpha\right|_\xi (y) dy, \\ I_2 &= \varepsilon_\alpha \int_{B_0(6\delta_\alpha)} |z_\alpha - y|^{1-n} \left| v_\alpha^{2^*} - B_\alpha^{2^*}\right| dy, \\ I_3 &= \left| \int_{B_0(6\delta_\alpha)} \mathcal{H}_{ij,\alpha}(z_\alpha, y)_p \left(\tilde{X}_\alpha(y) - \tilde{X}_\alpha(0)\right)^p v_\alpha^{2^*}(y) dy \right|, \\ I_4 &= \int_{B_0(6\delta_\alpha)} |z_\alpha - y|^{1-n} dy. \end{aligned} \tag{4.46}$$

We clearly have that

$$I_4 \le C\delta_\alpha. \tag{4.47}$$

By Claim 4.1 there exists $R_\alpha \to +\infty$ such that

$$\sup_{B_0(R_\alpha \mu_\alpha)} |v_\alpha - B_\alpha| = o(1)$$

as $\alpha \to +\infty$. Using Claim 4.2 we can therefore write

$$\begin{aligned} I_2 \le & \varepsilon_\alpha \int_{B_0(R_\alpha \mu_\alpha)} |z_\alpha - y|^{1-n} \left| v_\alpha - B_\alpha \right| (y) B_\alpha^{2^*-1}(y) dy \\ & + O\left(\varepsilon_\alpha \int_{B_0(6\delta_\alpha) \setminus B_0(R_\alpha \mu_\alpha)} |z_\alpha - y|^{1-n} B_\alpha^{2^*}(y) dy \right) \end{aligned}$$

so that in the end we get:

$$I_2 = o\left(\varepsilon_\alpha \theta_\alpha(z_\alpha)^{1-n}\right) \tag{4.48}$$

where $\theta_\alpha(z_\alpha)$ is as in (4.35). We let $\mathcal{H}_{ij}$ be the analogue of $\mathcal{H}_{ij,\alpha}$ in (4.44) for the fundamental solution of $\overrightarrow{\triangle}_\xi$ in $\mathbb{R}^n$, that is:

$$\mathcal{H}_{ij}(x,y)_p = \partial_i \mathcal{G}_j(x-y)_p + \partial_j \mathcal{G}_i(x-y)_p - \frac{2}{n}\xi_{ij} \sum_{k=1}^n \partial_k \mathcal{G}_k(x-y)_p, \tag{4.49}$$

where $\mathcal{G}_i$ is as in (4.31). Since, by (2.2), $\tilde{X}_\alpha$ has a limit in $C^2_{loc}(\mathbb{R}^n)$, we can write with (4.46) that:

$$\begin{aligned} I_3 \le & \left| \int_{B_0(s_\alpha)} \mathcal{H}_{ij,\alpha}(z_\alpha, y)_p y^k \partial_k \tilde{X}_\alpha(0)^p v_\alpha^{2^*}(y) dy \right| \\ & + C \int_{B_0(s_\alpha)} |z_\alpha - y|^{1-n} |y|^2 B_\alpha^{2^*}(y) dy \\ \le & \left| \int_{B_0(s_\alpha)} \mathcal{H}_{ij,\alpha}(z_\alpha, y)_p y^k \partial_k \tilde{X}_\alpha(0)^p v_\alpha^{2^*}(y) dy \right| + \mu_\alpha^2 \theta_\alpha(z_\alpha)^{1-n}. \end{aligned}$$

We now separate between two cases. First, assume that $|z_\alpha| = O(\mu_\alpha)$. Then one easily gets that

$$\left| \int_{B_0(s_\alpha)} \mathcal{H}_{ij,\alpha}(z_\alpha, y)_p y^k \partial_k \tilde{X}_\alpha(0)^p v_\alpha^{2^*}(y) dy \right| \le \mu_\alpha \theta_\alpha(z_\alpha)^{1-n}. \tag{4.50}$$

Next assume that

$$\frac{|z_\alpha|}{\mu_\alpha} \to +\infty \tag{4.51}$$

as $\alpha \to 0$. We write:

$$\begin{aligned} & \int_{B_0(s_\alpha)} \mathcal{H}_{ij,\alpha}(z_\alpha, y)_p y^k \partial_k \tilde{X}_\alpha(0)^p v_\alpha^{2^*}(y) dy \\ & \quad = \int_{B_0(s_\alpha)} \Big(\mathcal{H}_{ij,\alpha}(z_\alpha, y) - \mathcal{H}_{ij}(z_\alpha, y) \Big)_p y^k \partial_k \tilde{X}_\alpha(0)^p v_\alpha^{2^*}(y) dy \\ & \quad + \int_{B_0(s_\alpha)} \mathcal{H}_{ij}(z_\alpha, y)_p y^k \partial_k \tilde{X}_\alpha(0)^p v_\alpha^{2^*}(y) dy. \end{aligned}$$

Straightforward computations using (4.51) show that

$$\int_{B_0(s_\alpha)} \mathcal{H}_{ij}(z_\alpha, y)_p y^k \partial_k \tilde{X}_\alpha(0)^p v_\alpha^{2^*}(y) dy = o\left(\mu_\alpha \theta_\alpha(z_\alpha)^{1-n}\right), \tag{4.52}$$

where $\mathcal{H}_{ij}$ is as in (4.49). Let now

$$\Omega_\alpha = \left\{ y \in B_0(s_\alpha) \text{ st } |z_\alpha - y| \ge \frac{1}{2}|z_\alpha| \right\}.$$

On one hand there easily holds, using (4.51):

$$\begin{aligned}&\left|\int_{{}^c\Omega_\alpha}\Big(\mathcal{H}_{ij,\alpha}(z_\alpha,y)-\mathcal{H}_{ij}(z_\alpha,y)\Big)_p y^k\partial_k\tilde{X}_\alpha(0)^p v_\alpha^{2^*}(y)dy\right|\\&\qquad\le\left(\frac{\mu_\alpha}{\theta_\alpha(z_\alpha)}\right)^{n-1}\mu_\alpha\theta_\alpha(z_\alpha)^{1-n}=o\left(\mu_\alpha\theta_\alpha(z_\alpha)^{1-n}\right).\end{aligned}\tag{4.53}$$

On the other hand, by the definition of $\mathcal{G}_{\alpha,i}$ as in (8.14) below there holds, for any $y\in B_0(s_\alpha)\backslash\{z_\alpha\}$:

$$\mathcal{G}_{\alpha,i}(z_\alpha,y)=\mathcal{G}_i(z_\alpha-y)+U^{s_\alpha}_{i,z_\alpha}(y),$$

where $U^{s_\alpha}_{i,z_\alpha}$ is as in (8.13) below. The scaling arguments developed in Claim 8.4 below show that there holds:

$$s_\alpha^{n-2}U^{s_\alpha}_{i,z_\alpha}(s_\alpha\cdot)\to\tilde{U}_{0,i}\text{ in }C^1(\overline{B_0(1)}),\tag{4.54}$$

where $\tilde{U}_{0,i}$ satisfies:

$$\begin{cases}\overrightarrow{\triangle}_\xi\tilde{U}_{0,i}(y)=-\displaystyle\sum_{j=1}^{\frac{(n+1)(n+2)}{2}}K_j^0(\check{z})K_j^0(y) & B_0(1),\\ \nu^k\mathcal{L}_\xi\left(\tilde{U}_{0,i}\right)_{kl}(y)=-\nu^k\mathcal{L}_\xi\left(\mathcal{G}_i(\check{z}-\cdot)\right)_{kl}(y) & \partial B_0(1),\end{cases}$$

where we have let $\check{z}=\lim_{\alpha\to+\infty}\frac{z_\alpha}{s_\alpha}\in\overline{B_0(1)}$ and where $\left(K_j^0\right)_{1\le j\le(n+1)(n+2)/2}$ is an orthonormal basis for the L^2-scalar product of the set K_1 of conformal Killing 1-forms in $B_0(1)$ defined in (8.6). In the end, using (4.54), (8.10) below and Lebesgue's dominated convergence theorem there holds:

$$\begin{aligned}&\mu_\alpha^{-1}\beta_{\alpha,k}^{-1}|z_\alpha|^{n-1}\left|\int_{\Omega_\alpha}\Big(\mathcal{H}_{ij,\alpha}(z_\alpha,y)-\mathcal{H}_{ij}(z_\alpha,y)\Big)_p y^k\partial_k\tilde{X}_\alpha(0)^p v_\alpha^{2^*}(y)dy\right|\\&\quad=|\check{z}|^{n-1}\int_{\mathbb{R}^n}\left(\partial_i\tilde{U}_{0,j}+\partial_j\tilde{U}_{0,i}-\frac{2}{n}\xi_{ij}\sum_{k=1}^n\partial_k\tilde{U}_{0,k}\right)_p(0)\\&\qquad\qquad\times y^k\zeta_k^p\left(1+\frac{f_0(x_0)}{n(n-2)}|y|^2\right)^{-n}dy+o(1)\\&\quad=o(1),\end{aligned}\tag{4.55}$$

where $\beta_{\alpha,k}$ and ζ_k are as in (4.33) and (4.34). In (4.55), the notation $(\cdot)_p$ denotes the p-th coordinate of the 1-form considered. Gathering (4.52), (4.53) and (4.55) there thus holds, if $|z_\alpha|>>\mu_\alpha$:

$$\int_{B_0(s_\alpha)}\mathcal{H}_{ij,\alpha}(z_\alpha,y)_p y^k\partial_k\tilde{X}_\alpha(0)^p v_\alpha^{2^*}(y)dy=o\left(\mu_\alpha\theta_\alpha(z_\alpha)^{1-n}\right).\tag{4.56}$$

In the end, (4.50) and (4.56) together combine by writing that:

$$I_3=\eta\left(\frac{\mu_\alpha}{\theta_\alpha(z_\alpha)}\right)\mu_\alpha\theta_\alpha(z_\alpha)^{1-n}\tag{4.57}$$

for some continuous bounded η with $\eta(0) = 0$. We finally compute I_1. A Hölder inequality along with (4.25) gives, for any $p > 2$:

$$I_1 \le \left(\int_{B_0(6\delta_\alpha)} |z_\alpha - y|^{2-n} \frac{|\mathcal{L}_\xi Z_\alpha|_\xi^2}{v_\alpha^{2^*+1}}(y)dy \right)^{\frac{1}{p}} \times \left(\int_{B_0(6\delta_\alpha)} |z_\alpha - y|^{\frac{n-2-p(n-1)}{p-1}} |y|^{\frac{p}{p-1}} \left|\mathcal{L}_\xi Z_\alpha\right|_\xi (y)^{\frac{p-2}{p-1}} B_\alpha(y)^{\frac{2^*+1}{p-1}} dy \right)^{\frac{p-1}{p}}.$$

If $2 < p < 2n-2$, the second integral can be estimated with (4.7) as:

$$\int_{B_0(6\delta_\alpha)} |z_\alpha - y|^{\frac{n-2-p(n-1)}{p-1}} |y|^{\frac{p}{p-1}} \left|\mathcal{L}_\xi Z_\alpha\right|_\xi (y)^{\frac{p-2}{p-1}} B_\alpha(y)^{\frac{2^*+1}{p-1}} dy \le \mu_\alpha^{\frac{3n-2}{2(p-1)} + \frac{p+2-2n}{p-1}} \theta_\alpha(z_\alpha)^{\frac{n-2-p(n-1)}{p-1}},$$

so that using (4.27) to estimate the first integral yields in the end:

$$I_1 \le C\mu_\alpha \theta_\alpha(z_\alpha)^{1-n} \tag{4.58}$$

for some positive constant C. Gathering (4.47), (4.48), (4.57) and (4.58) in (4.45) and using (4.36) we obtain, since $\mu_\alpha \le \theta_\alpha(z_\alpha) \le O(\delta_\alpha)$ and by (4.40), that for $z_\alpha \in B_0(4\delta_\alpha)$:

$$\begin{aligned} \left|\mathcal{L}_\xi Z_\alpha\right|_\xi (z_\alpha) \le C \left(\varepsilon_\alpha \delta_\alpha^{1-n} + \mu_\alpha^{n-1} \delta_\alpha^{1-2n} \right) + C\varepsilon_\alpha \theta_\alpha(z_\alpha)^{1-n} \\ + C\mu_\alpha \theta_\alpha(z_\alpha)^{1-n}. \end{aligned} \tag{4.59}$$

We now use this to refine the estimate of I_1, where I_1 is as in (4.46). We assume that the sequence z_α belongs to $B_0(3\delta_\alpha)$. We therefore write:

$$I_1 \le \int_{B_0(4\delta_\alpha)} |z_\alpha - y|^{1-n} |y| \left|\mathcal{L}_\xi Z_\alpha\right|_\xi (y)dy + \delta_\alpha^{2-n} \int_{B_0(6\delta_\alpha)\backslash B_0(4\delta_\alpha)} \left|\mathcal{L}_\xi Z_\alpha\right|_\xi (y)dy,$$

and we use estimate (4.59) to compute the first integral and (4.28) to compute the second one. In the end these computations lead to:

$$I_1 \le C\mu_\alpha^{n-1}\delta_\alpha^{3-2n} + C\big(\mu_\alpha + \varepsilon_\alpha\big)\theta_\alpha(z_\alpha)^{3-n} \ln\left(2 + \frac{\delta_\alpha}{\theta_\alpha(z_\alpha)}\right)$$

which gives, using (4.40) and since $\theta_\alpha(z_\alpha) \le 4\delta_\alpha$:

$$I_1 \le C\mu_\alpha^{n-1}\delta_\alpha^{3-2n} + o\left(\mu_\alpha \theta_\alpha(z_\alpha)^{1-n}\right) + o\left(\varepsilon_\alpha \theta_\alpha(z_\alpha)^{1-n}\right). \tag{4.60}$$

Note that the factor $\ln\left(2 + \frac{\delta_\alpha}{\theta_\alpha(z_\alpha)}\right)$ in the computations above has to be taken into account only in dimension 3. In the end, gathering (4.47), (4.48), (4.57) and (4.60) in (4.45) yields, for any $z_\alpha \in B_0(3\delta_\alpha)$:

$$\begin{aligned} \left|\mathcal{L}_\xi \left(Z_\alpha - V_\alpha\right)\right|_\xi(z_\alpha) \le C\Big(\mu_\alpha^{n-1}\delta_\alpha^{1-2n} + \varepsilon_\alpha \delta_\alpha^{1-n}\Big) \\ + o\left(\varepsilon_\alpha \theta_\alpha(z_\alpha)^{1-n}\right) + \eta\left(\frac{\mu_\alpha}{\theta_\alpha(z_\alpha)}\right) \mu_\alpha \theta_\alpha(z_\alpha)^{1-n} \end{aligned} \tag{4.61}$$

where $\theta_\alpha(z_\alpha)$ is as in (4.35) and ε_α is as in (4.33). This proves (4.41).

We now use this pointwise control to obtain an estimate on ε_α. Let $0<\gamma<1$. We write that:

$$\begin{aligned}\int_{B_0(2\gamma\delta_\alpha)\backslash B_0(\gamma\delta_\alpha)} |\mathcal{L}_\xi V_\alpha|_\xi^2(y)dy \le 2\int_{B_0(2\gamma\delta_\alpha)\backslash B_0(\gamma\delta_\alpha)} |\mathcal{L}_\xi\left(Z_\alpha - V_\alpha\right)|_\xi^2(y) \\ +2\int_{B_0(2\gamma\delta_\alpha)\backslash B_0(\gamma\delta_\alpha)} |\mathcal{L}_\xi Z_\alpha|_\xi^2(y)dy.\end{aligned}$$

Thanks to (4.40) we can estimate the left-hand side with (4.38), while for the right-hand side, we estimate the first integral using (4.61) and the second one using (4.28). This yields:

$$\varepsilon_\alpha^2 \le C\gamma^{2n-2}\varepsilon_\alpha^2 + C\gamma^{-2n}\mu_\alpha^{2n-2}\delta_\alpha^{-2n} + o(\varepsilon_\alpha^2) + o(\mu_\alpha^2),$$

where C is some constant that does not depend on α nor on γ. Up to choosing γ small enough we therefore obtain (4.41). ☐

Note, as the proof of Claim 4.4 shows, that one could have obtained more easily a less precise estimate on I_3 defined in (4.46): indeed, requiring only the X_α to converge to X_0 in $C^1(M)$ would yield, for any $z_\alpha \in B_0(3\delta_\alpha)$:

$$I_3 \le \mu_\alpha \theta_\alpha(z_\alpha)^{1-n}$$

instead of (4.57). One would then subsequently obtain that

$$\varepsilon_\alpha \le C\left(\mu_\alpha^{n-1}\delta_\alpha^{-n} + \mu_\alpha\right) \tag{4.62}$$

in (4.42). This estimate is actually enough to conclude in dimensions 3 and 4 if the convergence of the X_α is in $C^1(M)$, see Claims 4.6 and 4.7 below. In dimensions $n\ge 5$ the $C^2(M)$ convergence is needed to push the analysis one step further but, as soon as $n\neq 6$, (4.62) remains precise enough to provide a suitable starting point to improve the estimates on $\mathcal{L}_\xi Z_\alpha$ up to second order. It turns out that if $n=6$ estimate (4.62) fails to be sufficient to perform the required second-order approximation of $\mathcal{L}_\xi Z_\alpha$ and that (4.42) is needed. See the computations leading to equation (4.105) below for more details.

In high dimensions Claim 4.4 is not enough to conclude. We now push the analysis one order further and estimate the difference $|\mathcal{L}_\xi\left(Z_\alpha - V_\alpha - \sum_{k=1}^n P_{\alpha,k}\right)|_\xi$. This time, unlike in Claim 4.4, controlling v_α by B_α with Claim 4.2 is not enough, and the precision of the approximation of v_α by the bubble B_α comes into play. Such a precision is unknown at this stage. In the next Claim we obtain a simultaneous description of the second order terms in the expansion of both v_α and $\mathcal{L}_\xi Z_\alpha$. We simultaneously carry out the ping-pong analysis to handle these unknown second-order terms. As stated in the previous paragraph, and as we will see in the last step of the proof of Proposition 3.3 – namely Claim 4.7 – Claim 4.4 is enough to conclude in dimensions 3 and 4. For the next Claim we will thus assume that $n\ge 5$.

Claim 4.5. *Assume $n\ge 5$. Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of* (2.1) *such that* (2.2) *and* (3.1) *hold. Let $(x_\alpha)_\alpha$ and $(\rho_\alpha)_\alpha$ be two sequences satisfying* (4.1) *and* (4.2) *and define v_α and Z_α as in* (4.3)*. Let $(\delta_\alpha)_\alpha$ be a sequence of positive numbers satisfying $\delta_\alpha = O(r_\alpha)$ and:*

$$\frac{\delta_\alpha}{\mu_\alpha}\to+\infty \quad \textit{and} \quad \begin{cases}\delta_\alpha = O\left(\mu_\alpha^{\frac{1}{2}}\right) & \textit{if } n=5,\\ \delta_\alpha = O\left(\mu_\alpha^{\frac{n-4}{n-2}}\right) & \textit{if } n\ge 6.\end{cases} \tag{4.63}$$

Then there holds

$$\varepsilon_\alpha + \mu_\alpha^2 \delta_\alpha^{-1} \beta_\alpha \le \frac{\mu_\alpha^{n-1}}{\delta_\alpha^n}, \tag{4.64}$$

where ε_α and β_α are as in (4.33), and for any $z_\alpha \in B_0(2\delta_\alpha)$ we have:

$$\left| \mathcal{L}_\xi Z_\alpha \right|_\xi (z_\alpha) \le C \left(\frac{\mu_\alpha}{\delta_\alpha} \right)^{n-1} \theta_\alpha(z_\alpha)^{-n}, \tag{4.65}$$

where $\theta_\alpha(z_\alpha)$ is as in (4.35).

Proof. The proof proceeds in several steps.

Step 1: *there holds, for any $z_\alpha \in B_0(2\delta_\alpha)$:*

$$\begin{aligned} \left| \mathcal{L}_\xi \Big(Z_\alpha - V_\alpha - \sum_{k=1}^n P_{\alpha,k} \Big) \right|_\xi (z_\alpha) \le C \Big(\mu_\alpha^{n-1} \delta_\alpha^{1-2n} + \varepsilon_\alpha \delta_\alpha^{1-n} + \mu_\alpha^2 \beta_\alpha \delta_\alpha^{-n} \Big) \\ + \Big(\varepsilon_\alpha + \mu_\alpha \beta_\alpha \Big) \mu_\alpha^{\frac{n-2}{2}} \|R_\alpha\|_\infty \theta_\alpha(z_\alpha)^{1-n} \\ + \mu_\alpha^2 \theta_\alpha(z_\alpha)^{1-n} + o\left(\varepsilon_\alpha \theta_\alpha(z_\alpha)^{1-n} \right), \end{aligned} \tag{4.66}$$

where we have let, in $B_0(8\rho_\alpha)$:

$$R_\alpha = v_\alpha - B_\alpha, \tag{4.67}$$

and the L^∞-norm of R_α is taken over $B_0(2\delta_\alpha)$.

To prove (4.66), we write once again a Green representation formula for $\overrightarrow{\triangle}_\xi$ in $B_0(3\delta_\alpha)$. By (4.32) and (4.4), there holds on $B_0(3\delta_\alpha)$:

$$\begin{aligned} \overrightarrow{\triangle}_\xi \left(Z_\alpha - V_\alpha - \sum_{k=1}^n P_{\alpha,k} \right)_i (y) = 2^* \xi^{kl} \partial_k \ln \varphi_\alpha \left(\mathcal{L}_\xi Z_\alpha \right)_{li} \\ + v_\alpha^{2^*}(y) \left(\tilde{X}_\alpha(y) - \tilde{X}_\alpha(0) - y^p \partial_p \tilde{X}_\alpha(0) \right)_i + \left(\tilde{Y}_\alpha \right)_i (y) \\ + \left(v_\alpha^{2^*}(y) - B_\alpha^{2^*}(y) \right) \left(\tilde{X}_\alpha(0) + y^p \partial_p \tilde{X}_\alpha(0) \right)_i, \end{aligned} \tag{4.68}$$

where φ_α is as in (3.8) so that we can write, for any $z_\alpha \in B_0(2\delta_\alpha)$:

$$\left| \mathcal{L}_\xi \Big(Z_\alpha - V_\alpha - \sum_{k=1}^n P_{\alpha,k} \Big) \right|_\xi (z_\alpha) \le C \left(J_B + J_1 + J_2 + J_3 + J_4 \right), \tag{4.69}$$

where we have let:

$$\begin{aligned} J_B &= \int_{\partial B_0(\frac{5}{2}\delta_\alpha)} |z_\alpha - y|^{1-n} \left| \mathcal{L}_\xi \left(Z_\alpha - V_\alpha - \sum_{k=1}^n P_{\alpha,k} \right) \right|_\xi (y) d\sigma(y), \\ J_1 &= \int_{B_0(3\delta_\alpha)} |z_\alpha - y|^{1-n} |y| \left| \mathcal{L}_\xi Z_\alpha \right|_\xi (y) dy, \\ J_2 &= \int_{B_0(3\delta_\alpha)} |z_\alpha - y|^{1-n} |y|^2 v_\alpha^{2^*}(y) dy, \\ J_3 &= \int_{B_0(3\delta_\alpha)} \left(\varepsilon_\alpha + \beta_\alpha |y| \right) |z_\alpha - y|^{1-n} \left| v_\alpha^{2^*} - B_\alpha^{2^*} \right| (y) dy, \\ J_4 &= \int_{B_0(3\delta_\alpha)} |z_\alpha - y|^{1-n} dy. \end{aligned} \tag{4.70}$$

We estimate J_B using Claim 4.3 (with a radius $\frac{1}{2}\delta_\alpha$): there holds that

$$J_B \le C\left(\mu_\alpha^{n-1}\delta_\alpha^{1-2n} + \varepsilon_\alpha \delta_\alpha^{1-n} + \mu_\alpha^2 \beta_\alpha \delta_\alpha^{-n}\right), \tag{4.71}$$

that

$$J_4 \le C\delta_\alpha, \tag{4.72}$$

and that

$$J_2 \le C\mu_\alpha^2 \theta_\alpha(z_\alpha)^{1-n}, \tag{4.73}$$

for some $C > 0$ independent of α. Using (4.41) and (4.36) it is easily seen that

$$J_1 \le o\left(\varepsilon_\alpha \theta_\alpha(z_\alpha)^{1-n}\right) + C\mu_\alpha^2 \theta_\alpha(z_\alpha)^{1-n} + C\mu_\alpha^{n-1}\delta_\alpha^{1-2n}, \tag{4.74}$$

where we have used that $\delta_\alpha^2 = O(\mu_\alpha)$ for any $n \ge 3$ due to (4.63). Finally there easily holds:

$$J_3 \le C\left(\varepsilon_\alpha + \mu_\alpha \beta_\alpha\right) \mu_\alpha^{\frac{n-2}{2}} \|R_\alpha\|_\infty \theta_\alpha(z_\alpha)^{1-n}. \tag{4.75}$$

Gathering (4.71), (4.72), (4.73), (4.74) and (4.75) in (4.69) we therefore obtain (4.66). Clearly there holds $\mu_\alpha^{\frac{n-2}{2}} \|R_\alpha\|_\infty \le 1$, but this control is not precise enough for our purposes. We now use the improved estimate (4.66) to obtain a better control on R_α.

Step 2: *there holds*

$$\|R_\alpha\|_\infty \le C M_\alpha \tag{4.76}$$

for some positive C, where

$$M_\alpha = \mu_\alpha^{\frac{n-2}{2}}\delta_\alpha^{2-n} + \mu_\alpha^{2-\frac{n}{2}} + \mu_\alpha^{5-\frac{3n}{2}}\delta_\alpha^{n+2} + \varepsilon_\alpha^2 \mu_\alpha^{1-\frac{3n}{2}}\delta_\alpha^{n+2} + \mu_\alpha^{5-\frac{3n}{2}}\delta_\alpha^n \beta_\alpha^2 \tag{4.77}$$

for all α.

By definition of R_α as in (4.67) and by (4.4), it satisfies :

$$\begin{aligned}\triangle_g R_\alpha = \left(\tilde{f}_\alpha - \tilde{f}_\alpha(0)\right) v_\alpha^{2^*-1} + \tilde{f}_\alpha(0)\left(v_\alpha^{2^*-1} - B_\alpha^{2^*-1}\right) - \tilde{h}_\alpha v_\alpha \\ + \left(\tilde{b}_\alpha + \left|\tilde{U}_\alpha + \varphi_\alpha^{2^*}\mathcal{L}_\xi Z_\alpha\right|_\xi^2\right) v_\alpha^{-2^*-1}.\end{aligned} \tag{4.78}$$

Let $(y_\alpha)_\alpha$ be a sequence of points of $B_0(2\delta_\alpha)$. First, if $\delta_\alpha \le |y_\alpha| \le 2\delta_\alpha$, there holds by (4.25):

$$|R_\alpha(y_\alpha)| = O\left(\mu_\alpha^{\frac{n-2}{2}}\delta_\alpha^{2-n}\right) = O(M_\alpha), \tag{4.79}$$

where M_α is as in (4.77). Assume now that $y_\alpha \in B_0(\delta_\alpha)$. A Green formula for $\triangle_\xi$ in $B_0(2\delta_\alpha)$ at y_α gives, using (4.63):

$$|R_\alpha(y_\alpha)| \le C\left(\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5\right) \tag{4.80}$$

for some positive C, where we have let:

$$\begin{aligned}
\mathcal{I}_1 &= \int_{B_0(2\delta_\alpha)} |y_\alpha - y|^{2-n} \left| \tilde{f}_\alpha(y) - \tilde{f}_\alpha(0) \right| v_\alpha^{2^*-1}(y) dy, \\
\mathcal{I}_2 &= \int_{B_0(2\delta_\alpha)} |y_\alpha - y|^{2-n} v_\alpha^{2^*-2}(y) \left| R_\alpha(y) \right| dy, \\
\mathcal{I}_3 &= \int_{B_0(2\delta_\alpha)} |y_\alpha - y|^{2-n} v_\alpha(y) dy, \\
\mathcal{I}_4 &= \int_{B_0(2\delta_\alpha)} |y_\alpha - y|^{2-n} \left(1 + |\mathcal{L}_\xi Z_\alpha|_\xi^2 (y) \right) dy, \\
\mathcal{I}_5 &= \int_{\partial B_0(2\delta_\alpha)} |y_\alpha - y|^{2-n} \left| R_\alpha(y) \right| d\sigma(y).
\end{aligned} \tag{4.81}$$

Using Claim 4.2 and (4.63) there holds that:

$$\mathcal{I}_5 \le C \mu_\alpha^{\frac{n-2}{2}} \delta_\alpha^{2-n}, \tag{4.82}$$

that

$$\mathcal{I}_1 \le C \mu_\alpha^{\frac{n}{2}} \theta_\alpha(y_\alpha)^{2-n}, \tag{4.83}$$

that

$$\mathcal{I}_3 \le C \mu_\alpha^{\frac{n-2}{2}} \theta_\alpha(y_\alpha)^{4-n}, \tag{4.84}$$

and that

$$\mathcal{I}_2 \le C \left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)} \right)^2 \|R_\alpha\|_\infty. \tag{4.85}$$

Using now (4.66), (4.36) and (4.37) there holds:

$$\begin{aligned}
\mathcal{I}_4 \le C \Bigg(& \mu_\alpha^{\frac{n-2}{2}} \delta_\alpha^{2-n} + \mu_\alpha^{5-\frac{3n}{2}} \delta_\alpha^{n+2} \\
& + \varepsilon_\alpha^2 \mu_\alpha^{1-\frac{3n}{2}} \delta_\alpha^{n+2} + \mu_\alpha^{5-\frac{3n}{2}} \delta_\alpha^n \beta_\alpha^2 + \mu_\alpha^{1-\frac{n}{2}} \|R_\alpha\|_\infty^2 \delta_\alpha^{n+2} \beta_\alpha^2 \Bigg).
\end{aligned} \tag{4.86}$$

Note that, by definition of R_α as in (4.67) and by Claims 4.1 and 4.2 there holds:

$$\|R_\alpha\|_\infty = o\left(\mu_\alpha^{1-\frac{n}{2}} \right). \tag{4.87}$$

In particular, combining (4.87) with (4.63) and (4.33) we obtain that for any $n \ge 6$:

$$\mu_\alpha^{1-\frac{n}{2}} \|R_\alpha\|_\infty^2 \delta_\alpha^{n+2} \beta_\alpha^2 = o\left(\|R_\alpha\|_\infty \right).$$

This gives, in (4.86):

$$\begin{aligned}
\mathcal{I}_4 \le C & \left(\mu_\alpha^{\frac{n-2}{2}} \delta_\alpha^{2-n} + \mu_\alpha^{5-\frac{3n}{2}} \delta_\alpha^{n+2} + \varepsilon_\alpha^2 \mu_\alpha^{1-\frac{3n}{2}} \delta_\alpha^{n+2} + \mu_\alpha^{5-\frac{3n}{2}} \delta_\alpha^n \beta_\alpha^2 \right) \\
& + o\left(\|R_\alpha\|_\infty \right),
\end{aligned} \tag{4.88}$$

so that gathering (4.82), (4.83), (4.84), (4.85) and (4.88) in (4.80) we obtain:

$$|R_\alpha(y_\alpha)| \le C \left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)} \right)^2 \|R_\alpha\|_\infty + C M_\alpha + o\left(\|R_\alpha\|_\infty \right), \tag{4.89}$$

where M_α is as in (4.77). Note that since $\theta_\alpha(y_\alpha) \geq \mu_\alpha$ there holds:

$$\left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)}\right)^2 \leq C \left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)}\right)^p \text{ for any } 1 \leq p \leq 2. \tag{4.90}$$

In particular the following estimate holds for any $1 \leq p \leq 2$:

$$|R_\alpha(y_\alpha)| \leq C \left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)}\right)^p \|R_\alpha\|_\infty + CM_\alpha + o\left(\|R_\alpha\|_\infty\right). \tag{4.91}$$

Plugging (4.91) into (4.81) we can refine the estimate on $\mathcal{I}_2$. We obtain, as long as $p < n-4$, that:

$$\begin{aligned} \mathcal{I}_2 &\leq C \left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)}\right)^2 \Big(M_\alpha + o(\|R_\alpha\|_\infty)\Big) + \left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)}\right)^{p+2} \|R_\alpha\|_\infty \\ &\leq CM_\alpha + o(\|R_\alpha\|_\infty) + \left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)}\right)^{p+2} \|R_\alpha\|_\infty. \end{aligned} \tag{4.92}$$

Using (4.92) in (4.80) it is easily seen that the estimate on R_α improves accordingly. Combining (4.92) and (4.91) we therefore obtain by induction that there exists some $p_0 > n-4$ such that

$$|R_\alpha(y_\alpha)| \leq C \left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)}\right)^{p_0} \|R_\alpha\|_\infty + CM_\alpha + o\left(\|R_\alpha\|_\infty\right).$$

Plugging once again this estimate into (4.81) and using this time that $p_0 > n-4$ we obtain the following estimate on $\mathcal{I}_2$:

$$\mathcal{I}_2 \leq C \left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)}\right)^{n-2} + CM_\alpha + o(\|R_\alpha\|_\infty),$$

which, combined with (4.82), (4.83), (4.84) and (4.88) in (4.80) and with (4.79) gives that:

$$|R_\alpha(y_\alpha)| \leq C \left(\frac{\mu_\alpha}{\theta_\alpha(y_\alpha)}\right)^{n-2} \|R_\alpha\|_\infty + CM_\alpha + o\left(\|R_\alpha\|_\infty\right) \tag{4.93}$$

for any sequence of points $y_\alpha \in B_0(2\delta_\alpha)$. To prove (4.76) we now proceed by contradiction. We assume that R_α is not identically zero and further assume that:

$$\lim_{\alpha \to +\infty} \frac{\|R_\alpha\|_\infty}{M_\alpha} \to +\infty. \tag{4.94}$$

We now let y_α be such that

$$|R_\alpha(y_\alpha)| = \|R_\alpha\|_\infty. \tag{4.95}$$

Then (4.93) shows that

$$|y_\alpha| = O(\mu_\alpha), \tag{4.96}$$

so that in particular $\frac{1}{C}\mu_\alpha \leq \theta_\alpha(y_\alpha) \leq C\mu_\alpha$ for some positive C. We introduce the functions

$$\tilde{R}_\alpha(x) = \mu_\alpha^{\frac{n-2}{2}} R_\alpha(\mu_\alpha x) \quad \text{and} \quad \tilde{v}_\alpha(x) = \mu_\alpha^{\frac{n-2}{2}} v_\alpha(\mu_\alpha x), \tag{4.97}$$

which are well-defined in $B_0(2\frac{r_\alpha}{\mu_\alpha})$. By (4.78) $\|\tilde{R}_\alpha\|_\infty^{-1}\tilde{R}_\alpha$ satisfies:

$$\begin{aligned}\triangle_g\left(\|\tilde{R}_\alpha\|_\infty^{-1}\tilde{R}_\alpha\right) &= \left(\tilde{f}_\alpha(\mu_\alpha\cdot)-\tilde{f}_\alpha(0)\right)\|\tilde{R}_\alpha\|_\infty^{-1}\tilde{v}_\alpha^{2^*-1}\\ &+\tilde{f}_\alpha(0)\|\tilde{R}_\alpha\|_\infty^{-1}\left(\tilde{v}_\alpha^{2^*-1}-\tilde{B}_\alpha^{2^*-1}\right)-\mu_\alpha^2\tilde{h}_\alpha(\mu_\alpha\cdot)\tilde{v}_\alpha\|\tilde{R}_\alpha\|_\infty^{-1}\\ &+\left(\mu_\alpha^{2n}\tilde{b}_\alpha(\mu_\alpha\cdot)+\left|\mu_\alpha^n\tilde{U}_\alpha+\mu_\alpha^n\varphi_\alpha^{2^*}\mathcal{L}_\xi Z_\alpha\right|_\xi^2\right)(\mu_\alpha\cdot)\|\tilde{R}_\alpha\|_\infty^{-1}\tilde{v}_\alpha^{-2^*-1}.\end{aligned}\tag{4.98}$$

With (4.97) assumption (4.94) becomes:

$$\|\tilde{R}_\alpha\|_\infty >> \left(\frac{\mu_\alpha}{\delta_\alpha}\right)^{n-2}+\mu_\alpha+\mu_\alpha^{4-n}\delta_\alpha^{n+2}+\varepsilon_\alpha^2\mu_\alpha^{-n}\delta_\alpha^{n+2}+\mu_\alpha^{4-n}\delta_\alpha^n\beta_\alpha^2,\tag{4.99}$$

so that there holds :

$$\begin{aligned}&\left|\tilde{f}_\alpha(\mu_\alpha y)-\tilde{f}_\alpha(0)\right|\|\tilde{R}_\alpha\|_\infty^{-1}+\mu_\alpha^2\tilde{h}_\alpha(\mu_\alpha y)\|\tilde{R}_\alpha\|_\infty^{-1}\\ &\qquad+\mu_\alpha^{2n}\left(\tilde{b}_\alpha(\mu_\alpha y)+\left|\mu_\alpha^n\tilde{U}_\alpha(\mu_\alpha y)\right|_\xi\right)^2\|\tilde{R}_\alpha\|_\infty^{-1}\to 0\end{aligned}\tag{4.100}$$

in $C^0_{loc}(\mathbb{R}^n)$. Independently, by Claim 4.1, there holds for some positive C:

$$\left|\|\tilde{R}_\alpha\|_\infty^{-1}\left(\tilde{v}_\alpha^{2^*-1}(y)-\tilde{B}_\alpha^{2^*-1}(y)\right)\right|\le C,\tag{4.101}$$

and using (4.66) we obtain:

$$\begin{aligned}\mu_\alpha^{2n}\left|\mathcal{L}_\xi Z_\alpha\right|_\xi^2(\mu_\alpha y)\|\tilde{R}_\alpha\|_\infty^{-1} &\le \|\tilde{R}_\alpha\|_\infty^{-1}\left(\varepsilon_\alpha^2\mu_\alpha^2+\mu_\alpha^4\beta_\alpha^2+\left(\frac{\mu_\alpha}{\delta_\alpha}\right)^{4n-2}+\mu_\alpha^6\right)\\ &\qquad+\beta_\alpha^2\mu_\alpha^{3+\frac{n}{2}}\|R_\alpha\|_\infty.\end{aligned}\tag{4.102}$$

Since $\mu_\alpha^{3+\frac{n}{2}}\|R_\alpha\|_\infty=O(\mu_\alpha^4)$, (4.102) becomes, using (4.99) and (4.63):

$$\mu_\alpha^{2n}\left|\mathcal{L}_\xi Z_\alpha\right|_\xi(\mu_\alpha y)\to 0\tag{4.103}$$

in $C^0_{loc}(\mathbb{R}^n)$ as $\alpha\to+\infty$. Gathering (4.100), (4.101) and (4.103) in (4.98) and by standard elliptic theory we obtain that $\|\tilde{R}_\alpha\|_\infty^{-1}\tilde{R}_\alpha\to\hat{R}_0$ in $C^1_{loc}(\mathbb{R}^n)$, where $\hat{R}_0$ is a solution of

$$\triangle_\xi\hat{R}_0=(2^*-1)\,f_0(x_0)U^{2^*-2}\hat{R}_0,\tag{4.104}$$

where $x_0=\lim_{\alpha\to+\infty}x_\alpha$ and U is as in Claim 4.1. In addition, estimate (4.93) gives, with assumption (4.94) and letting $\alpha\to+\infty$:

$$\left|\hat{R}_0(x)\right|\le C\,(1+|x|)^{2-n}\ \text{ for any } x\in\mathbb{R}^n.$$

In particular, $U^{2^*-2}\hat{R}_0^2\in L^1(\mathbb{R}^n)$. Since $\hat{R}_0$ satisfies (4.104), $\hat{R}_0$ is given by the Bianchi-Egnell classification result [5]. Since x_α is a critical point of v_α and by the definition of μ_α and B_α as in (4.8) and (4.20) we clearly have that $\hat{R}_0(0)=0$ and $\nabla\hat{R}_0(0)=0$, and this implies then that $\hat{R}_0\equiv 0$, see [5]. However, if we let $\hat{y}_\alpha=\frac{y_\alpha}{\mu_\alpha}$, where y_α is as in (4.95), there holds by (4.96) that $\hat{y}_\alpha=O(1)$, so that $\hat{y}_\alpha\to\hat{y}_0\in\mathbb{R}^n$ with $\hat{R}_0(\hat{y}_0)=1$. This contradicts (4.94) and concludes the proof of (4.76).

Step 3: *Conclusion.* We now plug (4.76) into estimate (4.66) and get:

$$\begin{aligned}\left|\mathcal{L}_\xi\Big(Z_\alpha - V_\alpha - \sum_{k=1}^n P_{\alpha,k}\Big)\right|_\xi (z_\alpha) &\le C\Big(\mu_\alpha^{n-1}\delta_\alpha^{1-2n} + \varepsilon_\alpha \delta_\alpha^{1-n} + \mu_\alpha^2 \beta_\alpha \delta_\alpha^{-n}\Big) \\ &+ o\Big(\varepsilon_\alpha \theta_\alpha(z_\alpha)^{1-n}\Big) + o\Big(\mu_\alpha^2 \beta_\alpha \theta_\alpha(z_\alpha)^{-n}\Big) + C\mu_\alpha^2 \theta_\alpha(z_\alpha)^{1-n}.\end{aligned} \tag{4.105}$$

To obtain (4.105) we crucially used estimate (4.42) of Claim 4.4 along with (4.63). To estimate ε_α and β_α we proceed as before. Let $0 < \gamma < 1$. We write with (4.28) and (4.63) that:

$$\begin{aligned}\int_{B_0(2\gamma\delta_\alpha)\backslash B_0(\gamma\delta_\alpha)} \left|\mathcal{L}_\xi V_\alpha + \sum_{k=1}^n \mathcal{L}_\xi P_{\alpha,k}\right|_\xi^2 (y) dy &\le C\mu_\alpha^{2n-2}\delta_\alpha^{2-3n}\gamma^{2-3n} \\ &+ C\int_{B_0(2\gamma\delta_\alpha)\backslash B_0(\gamma\delta_\alpha)} \left|\mathcal{L}_\xi\left(Z_\alpha - V_\alpha - \sum_{k=1}^n P_{\alpha,k}\right)\right|_\xi^2 (y).\end{aligned} \tag{4.106}$$

On one side, using (4.38) and (4.39), there holds, by (4.63):

$$\int_{B_0(2\gamma\delta_\alpha)\backslash B_0(\gamma\delta_\alpha)} \left|\mathcal{L}_\xi V_\alpha + \sum_{k=1}^n \mathcal{L}_\xi P_{\alpha,k}\right|_\xi^2 (y) dy \ge \frac{1}{C}\Big(\gamma^{2-n}\varepsilon_\alpha^2 \delta_\alpha^{2-n} + \gamma^{-n}\beta_\alpha^2 \mu_\alpha^4 \delta_\alpha^{-n}\Big)$$

for some positive constant C that does not depend on γ. On the other side there holds, using (4.105):

$$\begin{aligned}&\int_{B_0(2\gamma\delta_\alpha)\backslash B_0(\gamma\delta_\alpha)} \left|\mathcal{L}_\xi\left(Z_\alpha - V_\alpha - \sum_{k=1}^n P_{\alpha,k}\right)\right|_\xi^2 (y) \\ &\qquad \le C\Big(\gamma^n \mu_\alpha^{2n-2}\delta_\alpha^{2-3n} + \gamma^n \varepsilon_\alpha^2 \delta_\alpha^{2-n} + \gamma^n \beta_\alpha^2 \mu_\alpha^4 \delta_\alpha^{-n} + \gamma^{2-n}\mu_\alpha^4 \delta_\alpha^{2-n}\Big).\end{aligned}$$

Gathering the latter two results in (4.106) yields (4.64), up to choosing γ small enough, since there always holds:

$$\mu_\alpha^2 = O\left(\mu_\alpha^{n-1}\delta_\alpha^{-n}\right)$$

under assumption (4.63). Finally, (4.65) clearly follows from (4.36), (4.37), (4.64), (4.105) and (4.63) and this concludes the proof of the Claim. □

The next step of the proof consists in showing that estimate (4.65) actually holds up to the radius r_α. By Claim 4.5 above it is therefore enough to show that r_α satisfies (4.63).

Claim 4.6. *Assume that (2.2) holds. Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of (2.1) satisfying (3.1) and $(x_\alpha)_\alpha$, $(\rho_\alpha)_\alpha$ be two sequences satisfying (4.1) and (4.2). We assume that the assumptions of Theorem 2.1 hold. Then we have:*

- $r_\alpha = O\left(\mu_\alpha^{\frac{1}{2}}\right)$ *if* $3 \le n \le 5$ *and* $b_0 \not\equiv 0$,
- $r_\alpha = O\left(\mu_\alpha^{\frac{n-1}{n}}\right)$ *if* $n \ge 3$ *and* $X_0(x_0) \ne 0$,
- $r_\alpha = O\left(\mu_\alpha^{\frac{n-2}{n-1}}\right)$ *if* $n \ge 6$ *and* $\nabla f_0(x_0) \ne 0$,

- $r_\alpha = O\left(\mu_\alpha^{\frac{n-4}{n-2}}\right)$ *if* $n \geq 6$ *and*

$$\frac{n-2}{4(n-1)}R(g)(x_0) - h_0(x_0) - C(n)\frac{\triangle_g f_0(x_0)}{f_0(x_0)} > 0,$$

where $x_0 = \lim_{\alpha\to+\infty} x_\alpha$ *and* $C(n)$ *is some positive constant depending only on* n *given by* (4.120) *below. As a consequence, Claims 4.4 and 4.5 hold for* $\delta_\alpha = r_\alpha$ *and there holds:*

$$\left|\mathcal{L}_\xi Z_\alpha\right|_\xi (z_\alpha) \leq C\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-1}\theta_\alpha(z_\alpha)^{-n} \tag{4.107}$$

for all α *and all* $z_\alpha \in B_0(2r_\alpha)$.

Proof. We first assume that $3 \leq n \leq 5$ and $b_0 \not\equiv 0$. Let G_α be the Green function of the operator $\triangle_g + h_\alpha$ in M. Since $\triangle_g + h_0$ is coercive there holds, for any $x \in M$ and for some positive constant C:

$$u_\alpha(x) \geq \frac{1}{C}\int_M \left(f_\alpha u_\alpha^{2^*-1} + \frac{b_\alpha}{u_\alpha^{2^*+1}}\right)dv_g.$$

We have

$$f_\alpha u_\alpha^{2^*-1} + \frac{b_\alpha}{u_\alpha^{2^*+1}} \geq \frac{2\cdot 2^*}{2^*-1}b_\alpha\left(\frac{(2^*+1)b_\alpha}{(2^*-1)f_\alpha}\right)^{-\frac{2^*+1}{2\cdot 2^*}},$$

and since $b_0 \not\equiv 0$ there thus exists some positive ε_0 independent of α such that:

$$u_\alpha \geq \varepsilon_0. \tag{4.108}$$

By the definition of r_α as in (4.21) we therefore obtain

$$r_\alpha = O\left(\mu_\alpha^{\frac{1}{2}}\right). \tag{4.109}$$

If $3 \leq n \leq 4$, (4.107) is a consequence of Claim 4.4 with (4.109) since in these dimensions (4.109) implies $\mu_\alpha = O\left(\mu_\alpha^{n-1}r_\alpha^{-n}\right)$. If $n = 5$, (4.107) is a consequence of Claim 4.5 and of (4.109).

Assume next that $n \geq 3$ and $X_0(x_0) \neq 0$. This means that $\varepsilon_\alpha \not\to 0$ as $\alpha \to \infty$. We proceed by contradiction and assume that $r_\alpha >> \mu_\alpha^{\frac{n-1}{n}}$. We then let

$$\delta_\alpha = \min\left(r_\alpha, \mu_\alpha^{\frac{1}{2}}\right).$$

There holds $\delta_\alpha >> \mu_\alpha$. Estimate (4.42) of Claim 4.4 therefore applies and shows, since $\varepsilon_\alpha \not\to 0$, that:

$$\varepsilon_\alpha = O\left(\mu_\alpha^{n-1}\delta_\alpha^{-n}\right).$$

However, by definition of δ_α it is easily seen that $\delta_\alpha >> \mu_\alpha^{\frac{n-1}{n}}$, which yields a contradiction with $\varepsilon_\alpha \not\to 0$. Hence, $r_\alpha = O\left(\mu_\alpha^{\frac{n-1}{n}}\right)$ and (4.107) follows from Claim 4.5.

Assume now that $n \geq 6$ and that $X_0(x_0) = 0$. We let $(\delta_\alpha)_\alpha$ be some arbitrary sequence of positive numbers satisfying

$$\frac{\delta_\alpha}{\mu_\alpha} \to +\infty, \quad \delta_\alpha \leq r_\alpha \text{ and } \delta_\alpha = O\left(\mu_\alpha^{\frac{n-4}{n-2}}\right). \tag{4.110}$$

In view of the assumptions of Theorem 2.1, we only have two cases left to consider. First, assume that $\nabla f_0(x_0) \neq 0$. To extract informations on r_α as we previously

did for the $X_0(x_0) \neq 0$ case we start obtaining an estimate on $|\nabla \tilde{f}_\alpha(0)|_\xi$. To do this, we let $Y \in \mathbb{R}^n$ be some fixed vector of norm 1 and apply a Pohozaev identity to v_α on $B_0(\delta_\alpha)$, where δ_α satisfies (4.110). It writes as:

$$\begin{aligned}\int_{\partial B_0(\delta_\alpha)} \left(\frac{1}{2} Y^k \nu_k |\nabla v_\alpha|_\xi^2 - Y^k \partial_k v_\alpha \partial_\nu v_\alpha\right) d\sigma = \\ - \int_{B_0(\delta_\alpha)} Y^k \partial_k v_\alpha \tilde{h}_\alpha v_\alpha dy + \int_{B_0(\delta_\alpha)} Y^k \partial_k v_\alpha \tilde{f}_\alpha v_\alpha^{2^*-1} dy \\ + \int_{B_0(\delta_\alpha)} Y^k \partial_k v_\alpha \left(\tilde{b}_\alpha + \left|\tilde{U}_\alpha + \varphi_\alpha^{2^*} \mathcal{L}_\xi Z_\alpha\right|_\xi^2\right) v_\alpha^{-2^*-1} dy.\end{aligned} \tag{4.111}$$

Using the definition of r_α as in (4.21) and (4.110) we have that

$$\int_{\partial B_0(\delta_\alpha)} \left(\frac{1}{2} Y^k \nu_k |\nabla v_\alpha|_\xi^2 - Y^k \partial_k v_\alpha \partial_\nu v_\alpha\right) d\sigma = O\left(\mu_\alpha^{n-2} \delta_\alpha^{1-n}\right), \tag{4.112}$$

that

$$\int_{B_0(\delta_\alpha)} Y^k \partial_k v_\alpha \tilde{h}_\alpha v_\alpha dy = o(\mu_\alpha), \tag{4.113}$$

and that

$$\begin{aligned}\int_{B_0(\delta_\alpha)} Y^k \partial_k v_\alpha \tilde{f}_\alpha v_\alpha^{2^*-1} dy = O\left(\mu_\alpha^n \delta_\alpha^{-1-n}\right) + O(\mu_\alpha) \\ - \left(\frac{n-2}{2n} f_0(x_0)^{-\frac{n}{2}} K_n^{-n} + o(1)\right) Y^k \partial_k \tilde{f}_\alpha(0).\end{aligned} \tag{4.114}$$

By (4.110) we can use Claim 4.5 to estimate the last integral appearing in (4.111), thus obtaining:

$$\begin{aligned}\int_{B_0(\delta_\alpha)} Y^k \partial_k v_\alpha \left(\tilde{b}_\alpha + \left|\tilde{U}_\alpha + \varphi_\alpha^{2^*} \mathcal{L}_\xi Z_\alpha\right|_\xi^2\right) v_\alpha^{-2^*-1} dy \\ = O\left(\mu_\alpha^{n-2} \delta_\alpha^{1-n}\right) + o(\mu_\alpha).\end{aligned} \tag{4.115}$$

Gathering (4.112), (4.113), (4.114) and (4.115) in (4.111) we obtain that for any $Y \in \mathbb{R}^n, |Y|_\xi = 1$:

$$Y^k \partial_k \tilde{f}_\alpha(0) = O(\mu_\alpha) + \left(\mu_\alpha^{n-2} \delta_\alpha^{1-n}\right),$$

which gives in the end:

$$\left|\nabla \tilde{f}_\alpha\right|_\xi (0) = O(\mu_\alpha) + O\left(\mu_\alpha^{n-2} \delta_\alpha^{1-n}\right). \tag{4.116}$$

There holds $\left|\nabla \tilde{f}_\alpha\right|_\xi (0) \not\to 0$ as $\alpha \to +\infty$ since we assumed $\nabla f_0(x_0) \neq 0$. In particular, we obtain with (4.116) that there holds:

$$\delta_\alpha = O\left(\mu_\alpha^{\frac{n-2}{n-1}}\right) \tag{4.117}$$

for any sequence $(\delta_\alpha)_\alpha$ satisfying (4.110). Assume now by contradiction that there holds $r_\alpha >> \mu_\alpha^{\frac{n-4}{n-2}}$ as $\alpha \to +\infty$. The sequence $\mu_\alpha^{\frac{n-4}{n-2}}$ therefore satisfies (4.110) and hence with (4.117) there holds:

$$\mu_\alpha^{\frac{n-4}{n-2}} = 0\left(\mu_\alpha^{\frac{n-2}{n-1}}\right),$$

which is impossible since there holds $\frac{n-2}{n-1} > \frac{n-4}{n-2}$ for any $n \geq 6$. Hence, r_α satisfies $r_\alpha = O\left(\mu_\alpha^{\frac{n-4}{n-2}}\right)$ and hence r_α satisfies (4.110), which gives in the end, with (4.117):

$$r_\alpha = O\left(\mu_\alpha^{\frac{n-2}{n-1}}\right).$$

Estimate (4.107) now follows form Claim 4.5.

Finally, if there holds $X_0(x_0) = 0$ and $\nabla f_0(x_0) = 0$, we assume that

$$\frac{n-2}{4(n-1)} R(g)(x_0) - h_0(x_0) - C(n)\frac{\triangle_g f_0(x_0)}{f_0(x_0)} > 0, \tag{4.118}$$

where

$$C(n) = \frac{n-2}{2} K_n^{-n} \left(\int_{\mathbb{R}^n} \left(1 + \frac{|x|^2}{n(n-2)}\right)^{2-n} dx\right)^{-1}$$

and

$$K_n^{-n} = 2^{-n} \left(n(n-2)\right)^{\frac{n}{2}} \omega_n \tag{4.119}$$

is the energy of the standard bubble. Easy computations give that

$$C(n) = \frac{(n-2)(n-4)}{8(n-1)}. \tag{4.120}$$

We obtain the control on r_α as a direct consequence of the geometric condition (4.118). We write a Pohozaev identity on $B_0(r_\alpha)$: it writes as

$$\begin{aligned} &\int_{B_0(r_\alpha)} \left(x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha\right) \triangle_\xi v_\alpha dx \\ &\qquad = \int_{\partial B_0(r_\alpha)} \left(\frac{1}{2} r_\alpha |\nabla v_\alpha|_\xi^2 - \frac{n-2}{2} v_\alpha \partial_\nu v_\alpha - r_\alpha \left(\partial_\nu v_\alpha\right)^2\right) d\sigma. \end{aligned} \tag{4.121}$$

On one hand, using the definition of r_α, there holds:

$$\int_{\partial B_0(r_\alpha)} \left(\frac{1}{2} r_\alpha |\nabla v_\alpha|_\xi^2 - \frac{n-2}{2} v_\alpha \partial_\nu v_\alpha - r_\alpha \left(\partial_\nu v_\alpha\right)^2\right) d\sigma = O\left(\left(\frac{\mu_\alpha}{r_\alpha}\right)\right)^{n-2}. \tag{4.122}$$

On the other hand we can write, using (4.4), that

$$\int_{B_0(r_\alpha)} \left(x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha\right) \triangle_\xi v_\alpha dx = K_1 + K_2 + K_3, \tag{4.123}$$

where

$$\begin{aligned} K_1 &= -\int_{B_0(r_\alpha)} \left(x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha\right) \tilde{h}_\alpha(x) v_\alpha(x) dx, \\ K_2 &= \int_{B_0(r_\alpha)} \left(x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha\right) \tilde{f}_\alpha(x) v_\alpha^{2^*-1}(x) dx, \\ K_3 &= \int_{B_0(r_\alpha)} \left(x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha\right) \left(\tilde{b}_\alpha + \left|\tilde{U}_\alpha + \varphi_\alpha^{2^*} \mathcal{L}_\xi Z_\alpha\right|_\xi^2\right) v_\alpha^{-2^*-1} dx. \end{aligned} \tag{4.124}$$

Since $n \geq 6$ and by (4.24) we have, by easy computations and using Lebesgue's dominated convergence theorem, that:

$$K_1 = \frac{4(n-1)}{n-4} K_n^{-n} f_0(x_0)^{-\frac{n}{2}} \left(h_0(x_0) - \frac{n-2}{4(n-1)} R(g)(x_0)\right) \mu_\alpha^2 + o(\mu_\alpha^2) \tag{4.125}$$

where K_n^{-n} is as in (4.119). Using the definition of r_α and since $\tilde{b}_\alpha \geq 0$ we can write that:

$$K_3 \leq \int_{B_0(2R_\alpha \mu_\alpha)} \left(x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha \right) \left(\tilde{b}_\alpha + \left| \tilde{U}_\alpha + \varphi_\alpha^{2^*} \mathcal{L}_\xi Z_\alpha \right|_\xi^2 \right) v_\alpha^{-2^*-1},$$

where R_α is as in (4.23). We once again consider a sequence of radii $(\delta_\alpha)_\alpha$ satisfying (4.110). Using Claim 4.5 on $B_0(2\delta_\alpha)$ the latter becomes:

$$K_3 \leq o\left(\left(\frac{\mu_\alpha}{\delta_\alpha} \right) \right)^{n-2} + o(\mu_\alpha^2). \tag{4.126}$$

Finally, since $f_\alpha \to f_0$ in $C^2(M)$ as $\alpha \to +\infty$ we can write with (4.24) and the dominated convergence theorem that:

$$\begin{aligned} K_2 = & \frac{n-2}{2} f_0(x_0)^{-\frac{n}{2}} \frac{\triangle_g f_0(x_0)}{f_0(x_0)} K_n^{-n} \mu_\alpha^2 + o(\mu_\alpha^2) \\ & + o\left(\left(\frac{\mu_\alpha}{r_\alpha} \right) \right)^{n-2} + o\left(\mu_\alpha |\nabla \tilde{f}_\alpha|_\xi(0) \right). \end{aligned} \tag{4.127}$$

Using estimate (4.116) relative to the sequence $(\delta_\alpha)_\alpha$ to estimate $\left| \nabla \tilde{f}_\alpha \right|_\xi (0)$, combining it with (4.127), (4.122), (4.123), (4.125) and (4.126) and plugging everything into (4.121) we obtain:

$$\begin{aligned} & \left[\frac{n-2}{4(n-1)} R(g)(x_0) - h_0(x_0) - C(n) \frac{\triangle_g f_0(x_0)}{f_0(x_0)} \right] \mu_\alpha^2 \\ & \qquad \leq o(\mu_\alpha^2) + O\left(\frac{\mu_\alpha}{r_\alpha} \right)^{n-2} + o\left(\left(\frac{\mu_\alpha}{\delta_\alpha} \right) \right)^{n-2}. \end{aligned} \tag{4.128}$$

Assume by contradiction that $r_\alpha >> \mu_\alpha^{\frac{n-4}{n-2}}$. Then choosing $\delta_\alpha = \mu_\alpha^{\frac{n-4}{n-2}}$ yields a contradiction in (4.128) because of (4.118). Hence $r_\alpha = O\left(\mu_\alpha^{\frac{n-4}{n-2}} \right)$ and (4.107) follows from Claim 4.5. □

4.2. **Conclusion of the proof of Proposition 3.3.** Thanks to the asymptotic description of the defects of compactness of the sequences v_α and $\mathcal{L}_\xi Z_\alpha$ in the ball $B_0(r_\alpha)$ obtained in the previous subsection we now conclude the proof of Proposition 3.3. In the following Claim we show that $r_\alpha = \rho_\alpha$, hence that the sequence v_α equals the standard bubble at first order up to the radius ρ_α.

Claim 4.7. *There holds that, up to a subsequence:*

$$r_\alpha = \rho_\alpha. \tag{4.129}$$

In particular, $\rho_\alpha \to 0$ as $\alpha \to +\infty$ and Claims 4.4 and 4.5 apply with $\delta_\alpha = \rho_\alpha$ by Claim 4.6.

Proof. For any $x \in B_0(2)$, we let:

$$\hat{v}_\alpha(x) = \mu_\alpha^{1-\frac{n}{2}} r_\alpha^{n-2} v_\alpha(r_\alpha x). \tag{4.130}$$

Using (4.4) it is easily seen that $\hat{v}_\alpha$ satisfies:

$$\triangle_\xi \hat{v}_\alpha + r_\alpha^2 \hat{h}_\alpha \hat{v}_\alpha = \hat{f}_\alpha \hat{v}_\alpha^{2^*-1} + \frac{\hat{a}_\alpha}{\hat{v}_\alpha^{2^*+1}}, \tag{4.131}$$

where we have let:

$$\begin{aligned}
\hat{a}_\alpha(x) &= \frac{r_\alpha^{4n-2}}{\mu_\alpha^{2n-2}}\left(\hat{b}_\alpha + \left|\hat{U}_\alpha + \hat{\varphi}_\alpha^{2^*}\mathcal{L}_\xi Z_\alpha(r_\alpha\cdot)\right|_\xi^2\right)(x),\\
\hat{h}_\alpha(x) &= \tilde{h}_\alpha(r_\alpha x),\\
\hat{f}_\alpha(x) &= \tilde{f}_\alpha(r_\alpha x),\\
\hat{b}_\alpha(x) &= \tilde{b}_\alpha(r_\alpha x),\\
\hat{U}_\alpha(x) &= \tilde{U}_\alpha(r_\alpha x),\\
\hat{\varphi}_\alpha(x) &= \varphi_\alpha(r_\alpha x).
\end{aligned} \tag{4.132}$$

By definition of r_α, by Claim 4.2 and by (4.130) there holds, for some positive C that does not depend on α:

$$\frac{1}{C}\left(\left(\frac{\mu_\alpha}{r_\alpha}\right)^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)}|x|^2\right)^{1-\frac{n}{2}} \le \hat{v}_\alpha \le C\left(\frac{n(n-2)}{f_\alpha(x_\alpha)}\right)^{\frac{n-2}{2}}|x|^{2-n}. \tag{4.133}$$

Using estimate (4.107) and (4.133) one obtains that for any $x \in B_0(2)$,

$$\frac{\hat{a}_\alpha}{\hat{v}_\alpha^{2^*+1}}(x) \le C\left(\left(\frac{\mu_\alpha}{r_\alpha}\right)^2 + \frac{f_\alpha(x_\alpha)}{n(n-2)}|x|^2\right)^{\frac{n}{2}} \in L^\infty(B_0(2)). \tag{4.134}$$

By (4.131), (4.133), (4.134) and standard elliptic theory we get that

$$\hat{v}_\alpha \to \hat{v} \text{ in } C^1_{loc}(B_0(2)\backslash\{0\}) \tag{4.135}$$

as $\alpha \to +\infty$, and we have that, for $x \neq 0$:

$$\hat{v}(x) = \frac{\lambda_0}{|x|^{n-2}} + H(x), \tag{4.136}$$

where $\lambda_0 = \left(\frac{n(n-2)}{f_0(x_0)}\right)^{\frac{n-2}{2}}$ and H is a superharmonic function in $B_0(2)$. By Claim 4.2 there also holds $H \ge 0$ in $B_0(2)$.

We claim that $H(0) > 0$ if $r_\alpha < \rho_\alpha$. Indeed, by the definition of r_α as in (4.21), if we assume $r_\alpha < \rho_\alpha$ there exists a sequence $y_\alpha \in B_0(r_\alpha)$ such that there holds:

- either $v_\alpha(y_\alpha) = (1+\varepsilon)B_\alpha(y_\alpha)$,
- etiher $|\nabla v_\alpha(y_\alpha)|_\xi = (1+\varepsilon)|\nabla B_\alpha(y_\alpha)|_\xi$,
- or $\left(x^k\partial_k v_\alpha + \frac{n-2}{2}v_\alpha\right)(y_\alpha) = 0$.

Letting $\hat{y}_\alpha = \frac{y_\alpha}{r_\alpha}$, it is easily seen that each of the above three cases implies that either $H(\hat{y}_\alpha)$ or $\nabla H(\hat{y}_\alpha)$ are nonzero which implies, since H is superharmonic and nonnegative, that $H(0) > 0$.

We therefore show that $r_\alpha = \rho_\alpha$ by showing that $H(0) \le 0$. Note that since H is nonnegative and superharmonic this will actually show that $H(0) = 0$, and hence that H is everywhere zero. To do this, we let $0 < \delta < 1$ and write a Pohozaev identity for v_α on $B_0(\delta r_\alpha)$. We have:

$$\begin{aligned}
&\int_{B_0(\delta r_\alpha)}\left(x^k\partial_k v_\alpha + \frac{n-2}{2}v_\alpha\right)\triangle_\xi v_\alpha dx\\
&\qquad = \int_{\partial B_0(\delta r_\alpha)}\left(\frac{1}{2}\delta r_\alpha|\nabla v_\alpha|_\xi^2 - \frac{n-2}{2}v_\alpha\partial_\nu v_\alpha - \delta r_\alpha\left(\partial_\nu v_\alpha\right)^2\right)d\sigma.
\end{aligned} \tag{4.137}$$

By (4.135) and (4.136) there holds:

$$\begin{aligned}\int_{\partial B_0(r_\alpha)} &\left(\frac{1}{2}\delta r_\alpha |\nabla v_\alpha|_\xi^2 - \frac{n-2}{2} v_\alpha \partial_\nu v_\alpha - \delta r_\alpha \left(\partial_\nu v_\alpha\right)^2\right) d\sigma \\ &= \left(\frac{1}{2}(n-2)^2 \lambda_0 \omega_{n-1} H(0) + \varepsilon(\delta) + o(1)\right)\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2},\end{aligned} \tag{4.138}$$

where until the end of this subsection $\varepsilon(\delta)$ will denote some bounded quantity such that

$$\lim_{\delta \to 0} \varepsilon(\delta) = 0. \tag{4.139}$$

Independently, there holds

$$\int_{B_0(\delta r_\alpha)} \left(x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha\right) \triangle_\xi v_\alpha dx = L_1 + L_2 + L_3,$$

where

$$\begin{aligned} L_1 &= -\int_{B_0(r_\alpha)} \left(x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha\right) \tilde{h}_\alpha(x) v_\alpha(x) dx, \\ L_2 &= \int_{B_0(r_\alpha)} \left(x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha\right) \tilde{f}_\alpha(x) v_\alpha(x)^{2^*-1} dx, \\ L_3 &= \int_{B_0(r_\alpha)} \left(x^k \partial_k v_\alpha + \frac{n-2}{2} v_\alpha\right)\left(\tilde{b}_\alpha + \left|\tilde{U}_\alpha + \varphi_\alpha^{2^*} \mathcal{L}_\xi Z_\alpha\right|_\xi^2\right)(x) v_\alpha(x)^{-2^*-1} dx. \end{aligned}$$

Straightforward computations yield:

$$L_1 = \begin{cases} O\left(\mu_\alpha r_\alpha\right) & \text{if } n = 3, \\ O\left(\mu_\alpha^2 \ln\left(\frac{r_\alpha}{\mu_\alpha}\right)\right) & \text{if } n = 4, \\ \frac{4(n-1)}{n-4} K_n^{-n} f_0(x_0)^{-\frac{n}{2}} & \\ \quad \times \left(h_0(x_0) - \frac{n-2}{4(n-1)} R(g)(x_0)\right)\mu_\alpha^2 + o(\mu_\alpha^2) & \text{if } n \geq 5, \end{cases} \tag{4.140}$$

where K_n^{-n} is as in (4.119). Because of Claim 4.6, the conclusion of (4.116) is still valid when taking $\delta_\alpha = r_\alpha$. Hence, mimicking the computations that led to (4.127) we therefore obtain:

$$L_2 = \begin{cases} o\left(\frac{\mu_\alpha}{r_\alpha}\right) & \text{if } n = 3, \\ \frac{n-2}{2} f_0(x_0)^{-\frac{n}{2}} \frac{\triangle_g f_0(x_0)}{f_0(x_0)} K_n^{-n} \mu_\alpha^2 + o(\mu_\alpha^2) & \\ \qquad + o\left(\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2}\right) & \text{if } n \geq 4. \end{cases} \tag{4.141}$$

Using the definition of r_α as in (4.21) and the optimal estimate (4.107), we obtain that:

$$L_3 \leq o(\mu_\alpha^2) + o\left(\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2}\right). \tag{4.142}$$

We are now able to conclude the proof of Claim 4.7. We first treat the case where $3 \le n \le 5$. Then (4.138), (4.140), (4.141) and (4.142) in (4.137), along with Claim 4.6 give:

$$\left(\frac{1}{2}(n-2)^2\lambda_0\omega_{n-1}H(0)+\varepsilon(\delta)+o(1)\right)\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2} \le o\left(\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2}\right),$$

so that letting $\alpha \to +\infty$ and then $\delta \to 0$ yields:

$$H(0) \le 0.$$

Assume now that $n \ge 6$. Then (4.138), (4.140), (4.141) and (4.142) in (4.137) give:

$$\begin{aligned}&\frac{n-4}{4(n-1)}K_n^n f_0(x_0)^{\frac{n}{2}}\left(\frac{1}{2}(n-2)^2\lambda_0\omega_{n-1}H(0)+\varepsilon(\delta)+o(1)\right)\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2}\\ &\quad\le\left[h_0(x_0)-\frac{n-2}{4(n-1)}R(g)(x_0)+C(n)\frac{\triangle_g f_0(x_0)}{f_0(x_0)}\right]\mu_\alpha^2+o(\mu_\alpha^2),\end{aligned} \tag{4.143}$$

where K_n^{-n} is as in (4.119) and $C(n)$ is as in (4.120). We separate the proof for $n \ge 6$ between three cases. Assume first that $X_0(x_0) \ne 0$. Then by Claim 4.6 there holds $r_\alpha = O\left(\mu_\alpha^{\frac{n-1}{n}}\right)$ and then

$$\mu_\alpha^2 = o\left(\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2}\right),$$

so that (4.143) gives, taking the limit $\alpha \to \infty$ and then $\delta \to 0$:

$$H(0) \le 0.$$

Assume then that $\nabla f_0(x_0) \ne 0$. By Claim 4.6 there holds $r_\alpha = O\left(\mu_\alpha^{\frac{n-2}{n-1}}\right)$ and then

$$\mu_\alpha^2 = o\left(\left(\frac{\mu_\alpha}{r_\alpha}\right)^{n-2}\right).$$

Once again, (4.143) gives $H(0) \le 0$ taking the limit $\alpha \to \infty$ and $\delta \to 0$. Assume finally that

$$h_0(x_0) - \frac{n-2}{4(n-1)}R(g)(x_0) + C(n)\frac{\triangle_g f_0(x_0)}{f_0(x_0)} < 0.$$

In this case there holds

$$\begin{aligned}&\frac{1}{2}(n-2)^2\lambda_0\omega_{n-1}\frac{n-4}{4(n-1)}K_n^n f_0(x_0)^{\frac{n}{2}}H(0)\\ &\le \lim_{\alpha\to+\infty} r_\alpha^{n-2}\mu_\alpha^{4-n}\left[h_0(x_0)-\frac{n-2}{4(n-1)}R(g)(x_0)+C(n)\frac{\triangle_g f_0(x_0)}{f_0(x_0)}\right]\end{aligned}$$

and hence, once again, $H(0) \le 0$. This therefore concludes the proof of Claim 4.7 and shows that $r_\alpha = \rho_\alpha$. □

Remember that the definition of r_α in (4.21) depended on some fixed parameter $\varepsilon > 0$. Claim 4.7 shows that r_α actually does not depend on ε since $r_\alpha = \rho_\alpha$. This yields then:

$$\sup_{B_0(\rho_\alpha)}\left|\frac{v_\alpha}{B_\alpha}-1\right| = o(1)$$

as $\alpha \to +\infty$. Since by Claim 4.7 there holds $\rho_\alpha \to 0$ as $\alpha \to \infty$, this concludes the proof of Proposition 3.3.

5. Instability results

In this section we prove Theorem 2.3. We show that the assumptions of Theorem 2.1 are sharp by constructing blowing-up sequences of solutions of system (2.1). In dimensions 3 to 5 we construct such examples on the standard sphere. In dimensions $n \geq 6$ we construct them on closed manifolds of positive scalar curvature admitting a locally conformally flat pole and with no conformal Killing 1-forms. We adapt, in dimensions greater than 6, the constructions of Druet-Hebey [15] and distinguish between dimension 6 and dimensions $n \geq 7$.

A manifold (M, g) is said to have a conformally flat pole at $x_0 \in M$ if g is conformally flat in a neighborhood of x_0; locally conformally flat manifolds are such manifolds. Recall that a manifold (M, g) is said to have no conformal Killing 1-forms (or equivalently, no conformal Killing vector-fields) if any 1-form satisfying $\mathcal{L}_g X = 0$ in M is zero. Nontrivial conformal Killing 1-forms may be found on specific manifolds, but as shown in Beig-Chruściel-Schoen [2] they generically do not exist. Examples of manifolds of positive scalar curvature with a conformally flat pole and having no conformal Killing 1-forms are obtained by considering quotients of $\mathbb{S}^n$ by isometry groups acting freely and properly. For instance the projective space $P_n(\mathbb{R}) = \mathbb{S}^n/\{\pm 1\}$ is an example in any dimension $n \geq 3$.

5.1. Instability in dimension $3 \leq n \leq 5$.

We state our instability result in dimension 3, for the sake of clarity. However, such an easy construction can be carried out in dimensions 4 and 5 without additional difficulties.

Proposition 5.1. *Let $(\mathbb{S}^3, h)$ be the standard sphere. There exists a sequence $(U_\alpha, Y_\alpha)_\alpha$ respectively of smooth $(2,0)$-tensor fields and smooth 1-forms such that*

$$U_\alpha \longrightarrow U \text{ in } C^0(\mathbb{S}^3),$$

and

$$Y_\alpha \longrightarrow Y \text{ in } C^0(\mathbb{S}^3)$$

as $\alpha \to +\infty$, where $U \not\equiv 0$, and there exists a sequence $(u_\alpha, W_\alpha)_\alpha$, with $u_\alpha > 0$, satisfying:

$$\begin{cases} \triangle_h u_\alpha + \dfrac{3}{4} u_\alpha = \dfrac{3}{4} u_\alpha^5 + \dfrac{|U_\alpha + \mathcal{L}_h W_\alpha|_h^2}{u_\alpha^7}, \\ \overrightarrow{\triangle}_h W_\alpha = u_\alpha^6 X + Y_\alpha, \end{cases} \tag{5.1}$$

where $X \not\equiv 0$ is some fixed 1-form in $\mathbb{S}^3$ and $\max_M u_\alpha \to +\infty$ as $\alpha \to +\infty$.

Note that in view of Theorem 2.1, the coefficient X in (5.1) vanishes somewhere.

Proof. We consider $x_0 \in \mathbb{S}^3$ and consider spherical coordinates centered at x_0, which we will denote by (r, θ, ϕ). It is well known that in these coordinates the metric h takes the following form:

$$h(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2 r & 0 \\ 0 & 0 & \sin^2 r \sin^2 \theta \end{pmatrix},$$

where $r = d_h(x, x_0)$. Let $(\lambda_\alpha)_\alpha$, $\lambda_\alpha > 1$, be a sequence of positive numbers converging to 1 as $\alpha \to +\infty$. Let, for any $x \in \mathbb{S}^3$:

$$\varphi_\alpha(x) = \left(\lambda_\alpha^2 - 1\right)^{\frac{1}{4}} (\lambda_\alpha - r)^{-\frac{1}{2}}, \tag{5.2}$$

where $r = d_h(x_0, x)$. The functions φ_α satisfy:

$$\triangle_h \varphi_\alpha + \frac{3}{4}\varphi_\alpha = \frac{3}{4}\varphi_\alpha^5. \tag{5.3}$$

Let $0 < \delta < \frac{\pi}{2}$. For any α, let $Z_\alpha : (0, \pi) \to \mathbb{R}$ be the maximal solution of the following ODE:

$$Z_\alpha'' + 2\cot r Z_\alpha' + \left(1 - 2\cot^2 r\right) Z_\alpha = -\frac{3}{4}\varphi_\alpha^6(r), \tag{5.4}$$

satisfying $Z_\alpha(\pi/2) = 1$ and $Z_\alpha'(\pi/2) = 0$, where φ_α is as in (5.2). By (5.2) and standard ODE theory it is easily seen that, for any $\varepsilon > 0$, Z_α is uniformly bounded in α in $C^2([\varepsilon, \pi - \varepsilon])$. Let now $\eta \in C^\infty([0, \pi])$ be such that $\eta \not\equiv 0$ in $[0, \pi]$ and $\eta \equiv 0$ in $[0, \delta]$ and in $[\pi - \delta, \pi]$. We let, for any $x \in \mathbb{S}^3$:

$$W_\alpha(x) = \eta(r) Z_\alpha(r) \frac{\partial}{\partial r}, \tag{5.5}$$

where $r = d_h(x_0, x)$. By definition of η and Z_α, W_α is smooth in $\mathbb{S}^3$, it is zero in $B_{x_0}(\delta)$ and in $B_{-x_0}(\delta)$ and it is uniformly bounded in α in $C^2(\mathbb{S}^3)$. Straightforward computations using (5.4) and the fact that W_α is radial and only depends on r yield that W_α satisfies:

$$\overrightarrow{\triangle}_h W_\alpha = \varphi_\alpha^6 X_0 + Y_\alpha, \tag{5.6}$$

where we have let

$$X_0(x) = \eta(r) \frac{\partial}{\partial r}$$

and

$$Y_\alpha = -\frac{4}{3}\Big(2\eta'(r) Z_\alpha'(r) + \eta''(r) Z_\alpha(r) + 2\cot r \eta'(r) Z_\alpha(r)\Big) \frac{\partial}{\partial r}.$$

It is then easily seen that Y_α converges in $C^0(\mathbb{S}^3)$ to some smooth 1-form Y_0. It remains to define:

$$U_\alpha = -\mathcal{L}_h W_\alpha, \tag{5.7}$$

where W_α is as in (5.5). Here again, since W_α is uniformly bounded in $C^2(\mathbb{S}^3)$, U_α converges in $C^0(\mathbb{S}^3)$ to some symmetric traceless $(2,0)$-tensor U_0. Since we assumed that $Z_\alpha(\pi/2) = 1$ and $Z_\alpha'(\pi/2) = 0$ there holds:

$$\mathcal{L}_g W_{\alpha_{rr}}\left(\frac{\pi}{2}, \theta, \phi\right) = \frac{4}{3}\eta'\left(\frac{\pi}{2}\right),$$

so that it is always possible to choose η so as to have a nonzero U_0. Using (5.3), (5.6) and (5.7) we obtain in the end that $(\varphi_\alpha, W_\alpha)$ satisfy:

$$\begin{cases} \triangle_h \varphi_\alpha + \dfrac{3}{4}\varphi_\alpha = \dfrac{3}{4}\varphi_\alpha^5 + \dfrac{|U_\alpha + \mathcal{L}_h W_\alpha|_h^2}{\varphi_\alpha^7}, \\ \overrightarrow{\triangle}_h W_\alpha = \varphi_\alpha^6 X + Y_\alpha, \end{cases}$$

which concludes the proof of Proposition 5.1. $\square$

5.2. **Instability in dimensions $n \geq 7$.**

Proposition 5.2. *Let (M,g) be a closed manifold of dimension $n \geq 7$. We assume that (M,g) has a locally conformally flat pole, that (M,g) has positive scalar curvature and that (M,g) has no nontrivial conformal Killing 1-forms. There exist examples of smooth functions τ with $\nabla\tau \not\equiv 0$ and of traceless divergence-free tensor fields $U \not\equiv 0$ such that there exists sequences $(h_\alpha, u_\alpha, W_\alpha)_\alpha$ of smooth functions and smooth 1-forms in M satisfying:*

$$h_\alpha \underset{\alpha\to\infty}{\longrightarrow} \frac{n-2}{4(n-1)}\left(R(g) - |\nabla\tau|_g^2\right) \text{ in } C^0(M), \tag{5.8}$$

$u_\alpha > 0$, $\sup_M u_\alpha \to \infty$ and

$$\begin{cases} \triangle_g u_\alpha + h_\alpha u_\alpha = \dfrac{n(n-2)}{4} u_\alpha^{2^*-1} + \dfrac{\left|U + \mathcal{L}_g W_\alpha\right|_g^2}{u_\alpha^{2^*+1}}, \\ \overrightarrow{\triangle}_g W_\alpha = -\dfrac{n-1}{n} u_\alpha^{2^*} \nabla\tau. \end{cases} \tag{5.9}$$

Proof. Let $x_0 \in M$ be such that (M,g) has a locally conformally flat pole at x_0. Let $\varphi \in C^\infty(M)$, $\varphi > 0$ be such that $\tilde g := \varphi g$ is the round sphere metric in $\tilde B_{x_0}(\delta)$, where $\delta > 0$ and $\tilde B_{x_0}(\delta)$ is the ball of center x_0 and radius δ measured with respect to $\tilde g$. Let $\eta \in C^\infty(M)$ be a nonnegative function satisfying $\eta \equiv 1$ in $\tilde B_{x_0}(\frac{\delta}{2})$ and $\eta \equiv 0$ outside of $\tilde B_{x_0}(\delta)$. Let $\tau \in C^\infty(M)$ be a smooth function satisfying $\nabla\tau \equiv 0$ in $\tilde B_{x_0}(\delta)$ and U be a nonzero traceless divergence-free $(2,0)$-tensor field. We let (u_0, W_0), with $u_0 > 0$, be a smooth solution of the following system:

$$\begin{cases} \triangle_{\tilde g} u_0 + \dfrac{n-2}{4(n-1)}\left(S_{\tilde g} - \varphi^{2-2^*}|\nabla\tau|_g^2\right) u_0 = \dfrac{n(n-2)}{4} u_\alpha^{2^*-1} + \dfrac{1}{\varphi^{2\cdot 2^*}} \dfrac{\left|U + \mathcal{L}_g W_0\right|_g^2}{u_\alpha^{2^*+1}}, \\ \overrightarrow{\triangle}_g W_0 = -\dfrac{n-1}{n} \varphi^{2^*} u_0^{2^*} \nabla\tau. \end{cases} \tag{5.10}$$

By the conformal covariance property of the conformal laplacian finding such a (u_0, W_0) amounts to solve the following system:

$$\begin{cases} \triangle_g(\varphi u_0) + \dfrac{n-2}{4(n-1)}\left(S_g - |\nabla\tau|_g^2\right)(\varphi u_0) = \dfrac{n(n-2)}{4}(\varphi u_0)^{2^*-1} + \dfrac{\left|U + \mathcal{L}_g W_0\right|_g^2}{(\varphi u_0)^{2^*+1}}, \\ \overrightarrow{\triangle}_g W_0 = -\dfrac{n-1}{n}(\varphi u_0)^{2^*} \nabla\tau, \end{cases} \tag{5.11}$$

which is always possible as long as U is nonzero and $\|\nabla\tau\|_\infty + \|U\|_\infty \leq C$, where C is a positive constant that depends only on n and g. We refer for this to the existence result in Premoselli [35]. Let $(\lambda_\alpha)_\alpha$, $\lambda_\alpha > 1$, be a sequence of numbers such that $\lambda_\alpha \to 1$ as $\alpha \to \infty$. We let in the following:

$$\varphi_\alpha(x) = \left(\lambda_\alpha^2 - 1\right)^{\frac{n-2}{4}} (\lambda_\alpha - r)^{1-\frac{n}{2}}, \tag{5.12}$$

where $r = \cos d_{\tilde g}(x_0, x)$. Since $\tilde g$ is the round metric in $\tilde B_{x_0}(\delta)$ and by definition of η there holds:

$$\left(\triangle_{\tilde g} + \frac{n-2}{4(n-1)} S_{\tilde g}\right)(\eta\varphi_\alpha) = \frac{n(n-2)}{4}\eta\varphi_\alpha^{2^*-1} + 2\langle\tilde\nabla\eta, \tilde\nabla\varphi_\alpha\rangle_{\tilde g} + \varphi_\alpha \triangle_{\tilde g}\eta, \tag{5.13}$$

where $\tilde{\nabla}$ stands for the gradient operator for the metric $\tilde{g}$. For the sake of simplicity we let in the following:

$$E_n(u_0) = \frac{n-2}{4(n-1)}\varphi^{2-2^*}|\nabla\tau|_g^2 u_0. \tag{5.14}$$

Note that there holds:

$$E_n(u_0)(x_0) = 0. \tag{5.15}$$

Since M has no conformal Killing 1-forms, we can let Z_α be the unique 1-form satisfying in M:

$$\overrightarrow{\triangle}_g Z_\alpha = -\frac{n-1}{n}\left(u_0 + \eta\varphi_\alpha\right)^{2^*}\varphi^{2^*}\nabla\tau. \tag{5.16}$$

We define:

$$\begin{aligned} A_\alpha =& \frac{1}{\varphi^{2\cdot 2^*}}\left(\frac{\left|U + \mathcal{L}_g Z_\alpha\right|_g^2}{\left(u_0+\eta\varphi_\alpha\right)^{2^*+1}} - \frac{\left|U + \mathcal{L}_g W_0\right|_g^2}{u_0^{2^*+1}}\right) \\ &+ \left(\eta^{2^*-1} - \eta\right)\varphi_\alpha - 2\langle\tilde{\nabla}\eta, \tilde{\nabla}\varphi_\alpha\rangle_{\tilde{g}} - \varphi_\alpha\triangle_{\tilde{g}}\eta \end{aligned} \tag{5.17}$$

and

$$F_\alpha = \frac{n(n-2)}{4}\left[\left(u_0+\eta\varphi_\alpha\right)^{2^*-1} - {u_0}^{2^*-1} - \left(\eta\varphi_\alpha\right)^{2^*-1}\right] \tag{5.18}$$

We also let ψ_α be the unique solution in M of:

$$\triangle_{\tilde{g}}\psi_\alpha + \frac{n-2}{4(n-1)}S_{\tilde{g}}\psi_\alpha = \left(F_\alpha + A_\alpha\right). \tag{5.19}$$

Finally, we let

$$\tilde{u}_\alpha = u_0 + \varphi_\alpha + \psi_\alpha, \tag{5.20}$$

where φ_α is as in (5.12) and define W_α as the unique 1-form in M satisfying:

$$\overrightarrow{\triangle}_g W_\alpha = -\frac{n-1}{n}\varphi^{2^*}\tilde{u}_\alpha^{2^*}\nabla\tau. \tag{5.21}$$

It is easily seen that $(\tilde{u}_\alpha, W_\alpha)$ satisfies in M

$$\begin{cases} \triangle_{\tilde{g}}\tilde{u}_\alpha + \tilde{h}_\alpha\tilde{u}_\alpha = \dfrac{n(n-2)}{4}\tilde{u}_\alpha^{2^*-1} + \dfrac{1}{\varphi^{2\cdot 2^*}}\dfrac{\left|U+\mathcal{L}_g W_\alpha\right|_g^2}{\tilde{u}_\alpha^{2^*+1}}, \\ \overrightarrow{\triangle}_g W_\alpha = -\dfrac{n-1}{n}(\varphi\tilde{u}_\alpha)^{2^*}\nabla\tau, \end{cases} \tag{5.22}$$

where we have let

$$\begin{aligned} \tilde{h}_\alpha\tilde{u}_\alpha =& \frac{n-2}{4(n-1)}S_{\tilde{g}}\tilde{u}_\alpha + \frac{n(n-2)}{4}\left[\tilde{u}_\alpha^{2^*-1} - \left(u_0+\eta\varphi_\alpha\right)^{2^*-1}\right] - E_n(u_0) \\ &+ \frac{1}{\varphi^{2\cdot 2^*}}\left(\frac{\left|U+\mathcal{L}_g W_\alpha\right|_g^2}{\tilde{u}_\alpha^{2^*+1}} - \frac{\left|U+\mathcal{L}_g Z_\alpha\right|_g^2}{\left(u_0+\eta\varphi_\alpha\right)^{2^*+1}}\right), \end{aligned} \tag{5.23}$$

where Z_α and W_α are as in (5.16) and (5.21). In what follows we investigate the convergence of $\tilde{h}_\alpha$. First, we always have

$$\left|\left(u_0+\eta\varphi_\alpha\right)^{2^*} - {u_0}^{2^*}\right| \le C\mu_\alpha^{\frac{n-2}{2}} \text{ on } M\backslash\tilde{B}_{x_0}(\delta)$$

for some positive constant C, where we have let

$$\mu_\alpha^2 = \lambda_\alpha - 1, \tag{5.24}$$

so that with (5.10) and (5.16) there holds by standard elliptic theory, see Section 7 :

$$\left\| \mathcal{L}_g \left(Z_\alpha - W_0 \right) \right\|_{L^\infty(M)} = O\left(\mu_\alpha^{\frac{n-2}{2}} \right). \tag{5.25}$$

In particular this yields

$$\frac{1}{u_0^{2^*+1}} \left| \left| U + \mathcal{L}_g Z_\alpha \right|_g^2 - \left| U + \mathcal{L}_g W_0 \right|_g^2 \right| (x) \le C \mu_\alpha^{\frac{n-2}{2}} \tag{5.26}$$

for any $x \in M$, where C does not depend on α or on x. With (5.26) and (5.12) we can write that:

$$\left| U + \mathcal{L}_g Z_\alpha \right|_g^2 \left((u_0 + \eta \varphi_\alpha)^{-2^*-1} - {u_0}^{-2^*-1} \right) \le C \min\left(1, \varphi_\alpha\right)$$

in M. This then gives, with (5.26) and by the definition of A_α in (5.17) that:

$$|A_\alpha| \le O\left(\min(1, \varphi_\alpha)\right) + O\left(\mu_\alpha^{\frac{n-2}{2}} \right), \tag{5.27}$$

where μ_α is defined in (5.24). Independently, there holds with (5.18) that

$$|F_\alpha| \le O\left(\min(\varphi_\alpha, \varphi_\alpha^{2^*-2}) \right). \tag{5.28}$$

Let x_α be a sequence of points in M. By definition of φ_α as in (5.12) and by (5.24) there holds:

$$\frac{1}{C} \left(\frac{\mu_\alpha}{\mu_\alpha^2 + d_g(x_0, x_\alpha)^2} \right)^{\frac{n-2}{2}} \le \varphi_\alpha(x_\alpha) \le C \left(\frac{\mu_\alpha}{\mu_\alpha^2 + d_g(x_0, x_\alpha)^2} \right)^{\frac{n-2}{2}} \tag{5.29}$$

for some $C > 0$ independent of α. A Green formula using (5.19) gives:

$$\begin{aligned} |\psi_\alpha(x_\alpha)| &\le C \int_M d_g(x_\alpha, y)^{2-n} \left| F_\alpha \right|(y) dv_h(y) \\ &\quad + C \int_M d_g(x_\alpha, y)^{2-n} \left| A_\alpha \right|(y) dv_h(y). \end{aligned} \tag{5.30}$$

Using (5.27) there holds:

$$\begin{aligned} &\int_M d_g(x_\alpha, y)^{2-n} \left| A_\alpha \right|(y) dv_h(y) \\ &\qquad = O\left(\int_{d_g(x_0, y) \le \sqrt{\mu_\alpha}} d_g(x_\alpha, y)^{2-n} dv_h(y) \right) \\ &\qquad + O\left(\int_{d_g(x_0, y) \ge \sqrt{\mu_\alpha}} d_g(x_\alpha, y)^{2-n} \varphi_\alpha(y) dv_h(y) \right) \\ &\qquad + O\left(\mu_\alpha^{\frac{n-2}{2}} \right). \end{aligned} \tag{5.31}$$

With (5.29) we obtain:

$$\int_{d_g(x_0, y) \ge \sqrt{\mu_\alpha}} d_g(x_\alpha, y)^{2-n} \varphi_\alpha(y) dv_h(y) = O(\mu_\alpha) \tag{5.32}$$

so that there holds:

$$\int_M d_g(x_\alpha, y)^{2-n} \left| A_\alpha \right|(y) dv_h(y) = O(\mu_\alpha). \tag{5.33}$$

Using (5.28) there holds:

$$\begin{aligned}\int_M d_g(x_\alpha,y)^{2-n}\left|F_\alpha\right|(y)dv_h(y)&\\ &= O\left(\int_{d_g(x_0,y)\le\sqrt{\mu_\alpha}} d_g(x_\alpha,y)^{2-n}\varphi_\alpha^{2^*-2}dv_h(y)\right)\\ &+ O\left(\int_{d_g(x_0,y)\ge\sqrt{\mu_\alpha}} d_g(x_\alpha,y)^{2-n}\varphi_\alpha(y)dv_h(y)\right)\end{aligned}$$

and using (5.32) and (5.29) yields in the end:

$$\int_M d_g(x_\alpha,y)^{2-n}\left|F_\alpha\right|(y)dv_h(y) = O\left(\frac{\mu_\alpha}{\theta_\alpha(x_\alpha)}\right)^2 + O(\mu_\alpha), \tag{5.34}$$

where $\theta_\alpha(x_\alpha) = \left(\mu_\alpha^2 + d_g(x_\alpha,x_0)^2\right)^{\frac{1}{2}}$ and μ_α is as in (5.24). In the end, (5.33) and (5.34) in (5.30) give:

$$|\psi_\alpha(x_\alpha)| = O\left(\frac{\mu_\alpha}{\theta_\alpha(x_\alpha)}\right)^2 + O(\mu_\alpha). \tag{5.35}$$

Note that so far all the computations in the proof of Proposition 5.2 actually hold for any $n\ge 6$. We now use the assumption $n\ge 7$ to write that

$$\mu_\alpha^2\theta_\alpha(x_\alpha)^{-2} = \begin{cases} o\left(\mu_\alpha^{\frac{n-2}{2}}\theta_\alpha(x_\alpha)^{4-n}\right) & \text{if } \theta_\alpha(x_\alpha) << \sqrt{\mu_\alpha}\\ O(\mu_\alpha) & \text{if } \theta_\alpha(x_\alpha) \ge \frac{1}{C}\sqrt{\mu_\alpha}\end{cases}$$

so that (5.35) becomes, for $n\ge 7$:

$$|\psi_\alpha(x_\alpha)| = o\left(\mu_\alpha^{\frac{n-2}{2}}\theta_\alpha(x_\alpha)^{4-n}\right) + O(\mu_\alpha). \tag{5.36}$$

We can now investigate the convergence of $\tilde{h}_\alpha$ defined as in (5.23). First, note that (5.36) implies that

$$\frac{\psi_\alpha}{\tilde{u}_\alpha} \to 0 \text{ and } \tilde{u}_\alpha^{2^*-3}\psi_\alpha \to 0 \text{ in } C^0(M), \tag{5.37}$$

where $\tilde{u}_\alpha$ is as in (5.20). In particular (5.37) along with the definition of $\tilde{u}_\alpha$ show that $\sup_M \tilde{u}_\alpha \to +\infty$. There also holds by (5.37):

$$\frac{1}{\tilde{u}_\alpha}\left[\tilde{u}_\alpha^{2^*-1} - (u_0+\eta\varphi_\alpha)^{2^*-1}\right] = o(1) \text{ in } C^0(M). \tag{5.38}$$

Then, since u_0 is assumed to satisfy $E_n(u_0)(x_0)=0$, where E_n is defined in (5.14), we have as $\alpha\to+\infty$

$$\frac{E_n(u_0)}{\tilde{u}_\alpha} \to \frac{E_n(u_0)}{u_0} = \frac{n-2}{4(n-1)}\varphi^{2-2^*}\left|\nabla\tau\right|_g^2 \text{ in } C^0(M). \tag{5.39}$$

Finally, we write that

$$\begin{aligned}\frac{\left|U+\mathcal{L}_gW_\alpha\right|_g^2}{\tilde{u}_\alpha^{2^*+1}} - \frac{\left|U+\mathcal{L}_gZ_\alpha\right|_g^2}{(u_0+\eta\varphi_\alpha)^{2^*+1}} &= \left(\left|U+\mathcal{L}_gW_\alpha\right|_g^2 - \left|U+\mathcal{L}_gZ_\alpha\right|_g^2\right)\tilde{u}_\alpha^{-2^*-1}\\ &+ \left|U+\mathcal{L}_gZ_\alpha\right|_g^2\left(\tilde{u}_\alpha^{-2^*-1} - (u_0+\eta\varphi_\alpha)^{-2^*-1}\right),\end{aligned}$$

where W_α and Z_α are as in (5.21) and (5.16). With (5.20), (5.26) and (5.37) there holds:

$$\frac{1}{\tilde{u}_\alpha} |U + \mathcal{L}_g Z_\alpha|_g^2 \left(\tilde{u}_\alpha^{-2^*-1} - (u_0 + \eta \varphi_\alpha)^{-2^*-1} \right) = o(1) \text{ in } C^0(M). \tag{5.40}$$

We have, independently, by (5.16), (5.21) and (5.20):

$$\overrightarrow{\triangle}_g (W_\alpha - Z_\alpha) = -\frac{n-1}{n} \varphi^{2^*} \left(\tilde{u}_\alpha^{2^*} - (\tilde{u}_\alpha - \psi_\alpha)^{2^*} \right) \nabla \tau.$$

By standard elliptic theory (see Section 7) and (5.37) there holds, since $X \equiv 0$ in $B_{x_0}(\varepsilon)$:

$$\| \mathcal{L}_g (W_\alpha - Z_\alpha) \|_{L^\infty(M)} = o(1). \tag{5.41}$$

Gathering (5.38), (5.39), (5.40) and (5.41) in (5.23) we therefore obtain:

$$\tilde{h}_\alpha \to \frac{n-2}{4(n-1)} \left(S_{\tilde{g}} - \varphi^{2-2^*} |\nabla \tau|_g^2 \right) \text{ in } C^0(M). \tag{5.42}$$

We define, in the end:

$$u_\alpha = \varphi \tilde{u}_\alpha. \tag{5.43}$$

The conformal covariance property of the conformal laplacian gives that

$$\left(\triangle_g + \frac{n-2}{4(n-1)} R(g) \right) u_\alpha = \varphi^{2^*-1} \left(\triangle_{\tilde{g}} + \frac{n-2}{4(n-1)} S_{\tilde{g}} \right) \tilde{u}_\alpha,$$

and in the end we obtain that (u_α, W_α) satisfies:

$$\begin{cases} \triangle_g u_\alpha + h_\alpha u_\alpha = \dfrac{n(n-2)}{4} u_\alpha^{2^*-1} + \dfrac{|U + \mathcal{L}_g W_\alpha|_g^2}{u_\alpha^{2^*+1}}, \\ \overrightarrow{\triangle}_g W_\alpha = -\dfrac{n-1}{n} u_\alpha^{2^*} \nabla \tau \end{cases}$$

in M, where we have let:

$$h_\alpha u_\alpha = \frac{n-2}{4(n-1)} R(g) + \varphi^{2^*-2} \left(\tilde{h}_\alpha - \frac{n-2}{4(n-1)} S_{\tilde{g}} \right) u_\alpha$$

and where u_α and W_α are as in (5.43) and (5.21). The convergence in (5.42) gives in the end:

$$h_\alpha \to \frac{n-2}{4(n-1)} \left(R(g) - |\nabla \tau|_g^2 \right),$$

which concludes the proof of Proposition 5.2. ☐

Notice that the same argument that led to the existence of (u_0, W_0) satisfying (5.10) shows that the limiting system (5.9) when h_α is replaced by its limit in (5.8) possesses nontrivial solutions.

5.3. **Instability in dimension 6.**

Proposition 5.3. *Let (M,g) be a closed manifold of dimension 6. We assume that (M,g) has a locally conformally flat pole, that (M,g) has positive scalar curvature and that (M,g) has no nontrivial conformal Killing 1-forms. There exist examples of smooth functions τ, h with $\nabla \tau \not\equiv 0$, $h > 6$ and of traceless and divergence-free tensor fields $U \not\equiv 0$ in M such that there exists sequences $(h_\alpha, u_\alpha, W_\alpha)_\alpha$ of smooth functions and smooth 1-forms in M satisfying:*

$$h_\alpha \underset{\alpha \to \infty}{\longrightarrow} h \text{ in } C^0(M),$$

$u_\alpha > 0$, $\sup_M u_\alpha \to \infty$ *and*

$$\begin{cases} \triangle_g u_\alpha + h_\alpha u_\alpha = 6u_\alpha^2 + \dfrac{\left|U + \mathcal{L}_g W_\alpha\right|_g^2}{u_\alpha^4}, \\ \overrightarrow{\triangle}_g W_\alpha = -\dfrac{5}{6} u_\alpha^3 \nabla\tau. \end{cases}$$

Proof. As before, we let $x_0 \in M$ be such that (M,g) has a locally conformally flat pole at x_0, $\varphi \in C^\infty(M)$, $\varphi > 0$ be such that $\tilde{g} := \varphi g$ is the round sphere metric in $\tilde{B}_{x_0}(\delta)$ (the ball being taken with respect to the metric $\tilde{g}$) and we let $\eta \in C^\infty(M)$ be a nonnegative function satisfying $\eta \equiv 1$ in $\tilde{B}_{x_0}(\frac{\delta}{2})$ and $\eta \equiv 0$ outside of $\tilde{B}_{x_0}(\delta)$. Let $(\lambda_\alpha)_\alpha$, $\lambda_\alpha > 1$, be a sequence of numbers such that $\lambda_\alpha \to 1$ as $\alpha \to \infty$. We let τ be a smooth function in M which is constant in $\tilde{B}_{x_0}(\delta)$ and U a nonzero traceless divergence free tensor field in M. Since (M,g) has no nontrivial conformal Killing 1-forms we can let (u_0, W_0), with $u_0 > 0$, be a solution of

$$\begin{cases} \triangle_{\tilde{g}} u_0 + \dfrac{1}{5} S_{\tilde{g}} u_0 = -6u_\alpha^2 + \dfrac{1}{\varphi^6} \dfrac{\left|U + \mathcal{L}_g W_0\right|_g^2}{u_\alpha^4}, \\ \overrightarrow{\triangle}_g W_0 = -\dfrac{5}{6} \varphi^3 u_0^3 \nabla\tau. \end{cases} \tag{5.44}$$

Here again, the conformal covariance property of the conformal laplacian reduces the resolution of (5.44) to the resolution of the following system:

$$\begin{cases} \triangle_g (\varphi u_0) + \dfrac{1}{5} S_g (\varphi u_0) = -6(\varphi u_0)^2 + \dfrac{\left|U + \mathcal{L}_g W_0\right|_g^2}{(\varphi u_0)^4}, \\ \overrightarrow{\triangle}_g W_0 = -\dfrac{5}{6} (\varphi u_0)^3 \nabla\tau. \end{cases} \tag{5.45}$$

The arguments developed in Dahl-Humbert-Gicquaud [13], Proposition 2.1, yield the existence of such an (u_0, W_0), at least when $\|\nabla\tau\|_\infty$ is small enough. Let $(\lambda_\alpha)_\alpha$, $\lambda_\alpha > 1$, be a sequence of numbers such that $\lambda_\alpha \to 1$ as $\alpha \to \infty$. We let in the following:

$$\varphi_\alpha(x) = \left(\lambda_\alpha^2 - 1\right) \left(\lambda_\alpha - r\right)^{-2}, \tag{5.46}$$

where $r = \cos d_{\tilde{g}}(x_0, x)$. Since $\tilde{g}$ is the round metric in $\tilde{B}_{x_0}(\delta)$ and by definition of η there holds:

$$\left(\triangle_{\tilde{g}} + \frac{1}{5} S_{\tilde{g}}\right) (\eta\varphi_\alpha) = 6\eta\varphi_\alpha^2 + 2\langle\tilde{\nabla}\eta, \tilde{\nabla}\varphi_\alpha\rangle_{\tilde{g}} + \varphi_\alpha \triangle_{\tilde{g}} \eta, \tag{5.47}$$

where $\tilde{\nabla}$ stands for the gradient operator for the metric $\tilde{g}$. We let Z_α be the unique 1-form satisfying in M:

$$\overrightarrow{\triangle}_g Z_\alpha = -\frac{5}{6} \left(u_0 + \eta\varphi_\alpha\right)^3 \varphi^3 \nabla\tau. \tag{5.48}$$

We define:

$$\begin{aligned} A_\alpha = {} & \frac{1}{\varphi^6} \left(\frac{\left|U + \mathcal{L}_g Z_\alpha\right|_g^2}{\left(u_0 + \eta\varphi_\alpha\right)^4} - \frac{\left|U + \mathcal{L}_g W_0\right|_g^2}{u_0^4} \right) \\ & + \left(\eta^2 - \eta\right) \varphi_\alpha - 2\langle\tilde{\nabla}\eta, \tilde{\nabla}\varphi_\alpha\rangle_{\tilde{g}} - \varphi_\alpha \triangle_{\tilde{g}} \eta. \end{aligned} \tag{5.49}$$

We also let ψ_α be the unique solution in M of:

$$\triangle_{\tilde{g}} \psi_\alpha + \frac{1}{5} S_{\tilde{g}} \psi_\alpha = \varphi^2 A_\alpha. \tag{5.50}$$

Finally, we let

$$\tilde{u}_\alpha = u_0 + \varphi_\alpha + \psi_\alpha, \tag{5.51}$$

where φ_α is as in (5.46) and define W_α as the unique 1-form in M satisfying:

$$\overrightarrow{\triangle}_g W_\alpha = -\frac{5}{6}\varphi^3 \tilde{u}_\alpha^3 \nabla \tau. \tag{5.52}$$

It is easily seen that $(\tilde{u}_\alpha, W_\alpha)$ satisfies in M

$$\begin{cases} \triangle_{\tilde{g}} \tilde{u}_\alpha + \tilde{h}_\alpha \tilde{u}_\alpha = 6\tilde{u}_\alpha^2 + \dfrac{1}{\varphi^6} \dfrac{|U + \mathcal{L}_g W_\alpha|_g^2}{\tilde{u}_\alpha^4}, \\ \overrightarrow{\triangle}_g W_\alpha = -\dfrac{5}{6}(\varphi \tilde{u}_\alpha)^3 \nabla \tau, \end{cases} \tag{5.53}$$

where we have let

$$\begin{aligned} \tilde{h}_\alpha \tilde{u}_\alpha = & \frac{1}{5} S_{\tilde{g}} \tilde{u}_\alpha + 6\left(\tilde{u}_\alpha^2 - \varphi_\alpha^2\right) \\ & + \frac{1}{\varphi^6}\left(\frac{|U + \mathcal{L}_g W_\alpha|_g^2}{\tilde{u}_\alpha^4} - \frac{|U + \mathcal{L}_g Z_\alpha|_g^2}{(u_0 + \eta\varphi_\alpha)^4}\right), \end{aligned} \tag{5.54}$$

where Z_α and W_α are as in (5.48) and (5.52). In what follows we investigate the convergence of $\tilde{h}_\alpha$. First, we always have

$$\left|(u_0 + \eta\varphi_\alpha)^3 - {u_0}^3\right| \le C\mu_\alpha^2 \text{ on } M \backslash \tilde{B}_{x_0}(\delta)$$

for some positive constant C, where we have let

$$\mu_\alpha^2 = \lambda_\alpha - 1, \tag{5.55}$$

so that with (5.44) and (5.48) there holds by standard elliptic theory, see Section 7 :

$$\left\|\mathcal{L}_g\left(Z_\alpha - W_0\right)\right\|_{L^\infty(M)} = O\left(\mu_\alpha^2\right). \tag{5.56}$$

In particular this yields

$$\frac{1}{u_0^4}\left||U + \mathcal{L}_g Z_\alpha|_g^2 - |U + \mathcal{L}_g W_0|_g^2\right|(x) \le C\mu_\alpha^2 \tag{5.57}$$

for any $x \in M$, where C does not depend on α or on x. With (5.57) and (5.46) we can write that:

$$|U + \mathcal{L}_g Z_\alpha|_g^2\left((u_0 + \eta\varphi_\alpha)^{-4} - {u_0}^{-4}\right) \le C\min\left(1, \varphi_\alpha\right)$$

in M. This then gives, with (5.57) and by the definition of A_α in (5.49) that:

$$|A_\alpha| \le O\left(\min(1, \varphi_\alpha)\right) + O\left(\mu_\alpha^2\right), \tag{5.58}$$

where μ_α is defined in (5.55). Let x_α be a sequence of points in M. By definition of φ_α as in (5.46) and by (5.55) there holds:

$$\frac{1}{C}\left(\frac{\mu_\alpha}{\mu_\alpha^2 + d_g(x_0, x_\alpha)^2}\right)^2 \le \varphi_\alpha(x_\alpha) \le C\left(\frac{\mu_\alpha}{\mu_\alpha^2 + d_g(x_0, x_\alpha)^2}\right)^2 \tag{5.59}$$

for some $C > 0$ independent of α. A Green formula using (5.50) gives:

$$|\psi_\alpha(x_\alpha)| \le C\int_M d_g(x_\alpha, y)^{-4}\,|A_\alpha|\,(y)\,dv_h(y). \tag{5.60}$$

The computations that led to (5.33) apply here and yield:

$$\int_M d_g(x_\alpha, y)^{-4} \left| A_\alpha \right| (y) dv_h(y) = O(\mu_\alpha). \tag{5.61}$$

In the end, (5.61) and (5.60) give:

$$|\psi_\alpha(x_\alpha)| = O(\mu_\alpha). \tag{5.62}$$

In particular (5.62) along with the definition of $\tilde{u}_\alpha$ in (5.51) show that $\sup_M \tilde{u}_\alpha \to +\infty$. We now write that

$$\begin{aligned}\frac{\left|U + \mathcal{L}_g W_\alpha\right|_g^2}{\tilde{u}_\alpha^4} - \frac{\left|U + \mathcal{L}_g Z_\alpha\right|_g^2}{\left(u_0 + \eta\varphi_\alpha\right)^4} &= \left(\left|U + \mathcal{L}_g W_\alpha\right|_g^2 - \left|U + \mathcal{L}_g Z_\alpha\right|_g^2\right) \tilde{u}_\alpha^{-4} \\ &\quad + \left|U + \mathcal{L}_g Z_\alpha\right|_g^2 \left(\tilde{u}_\alpha^{-4} - \left(u_0 + \eta\varphi_\alpha\right)^{-4}\right),\end{aligned}$$

where W_α and Z_α are as in (5.52) and (5.48). Mimicking the computations that led to (5.41) and using (5.62) we obtain:

$$\frac{1}{\tilde{u}_\alpha} \left|\mathcal{L}_g Z_\alpha\right|_g^2 \left(\tilde{u}_\alpha^{-4} - \left(u_0 + \eta\varphi_\alpha\right)^{-4}\right) = o(1) \text{ in } C^0(M), \tag{5.63}$$

and

$$\left\|\mathcal{L}_g \left(W_\alpha - Z_\alpha\right)\right\|_{L^\infty(M)} = o(1). \tag{5.64}$$

Finally, using (5.62) we have that

$$u_\alpha^2 - \varphi_\alpha^2 = 2u_0 u_\alpha - u_0^2 + o(1) + o(u_\alpha). \tag{5.65}$$

Gathering (5.63) and (5.64) in (5.54) we obtain:

$$\tilde{h}_\alpha \to \frac{1}{5} S_{\tilde{g}} + 12 u_0 \text{ in } C^0(\mathbb{S}^6). \tag{5.66}$$

We finally define:

$$u_\alpha = \varphi \tilde{u}_\alpha. \tag{5.67}$$

The conformal covariance property of the conformal laplacian gives that

$$\left(\triangle_g + \frac{1}{5} R(g)\right) u_\alpha = \varphi^2 \left(\triangle_{\tilde{g}} + \frac{1}{5} S_{\tilde{g}}\right) \tilde{u}_\alpha,$$

and we obtain that (u_α, W_α) satisfies:

$$\begin{cases} \triangle_g u_\alpha + h_\alpha u_\alpha = 6u_\alpha^2 + \dfrac{\left|U + \mathcal{L}_g W_\alpha\right|_g^2}{u_\alpha^4}, \\ \overrightarrow{\triangle}_g W_\alpha = -\dfrac{5}{6} u_\alpha^3 \nabla\tau \end{cases}$$

in M, where we have let:

$$h_\alpha u_\alpha = \frac{1}{5} R(g) + \varphi^2 \left(\tilde{h}_\alpha - \frac{1}{5} S_{\tilde{g}}\right) \tilde{u}_\alpha$$

and where u_α and W_α are as in (5.67) and (5.52). The convergence in (5.66) gives in the end:

$$h_\alpha \to \frac{1}{5} R(g) + 12\varphi^2 u_0,$$

which concludes the proof of Proposition 5.3. □

Here again, the limiting system obtained by replacing h_α by h in (5.9) possesses nontrivial solutions provided $\|\nabla\tau\|_\infty+\|U\|_\infty$ is small enough. It is a straightforward consequence of the arguments developed in Premoselli [35].

6. A Harnack inequality

We prove in this section a Harnack inequality for solutions of the Einstein-Lichnerowicz equation which was used throughout the paper :

Proposition 6.1. *Let a, f, h be smooth functions on $\overline{B_0(2)} \subset \mathbb{R}^n$ and let $u \in C^2(B_0(2))$ be a positive solution in $B_0(2)$ of*

$$\Delta_\xi u + hu = fu^{2^*-1} + \frac{a}{u^{2^*+1}} \ .$$

We assume that $f \geq 0$ and $a \geq 0$, $a \not\equiv 0$ and that

$$\|u\|_{L^\infty(B_0(2))} \leq M$$

for some positive M. Then there exists $C > 0$ depending only on $\|h\|_{L^\infty(B_0(2))}$, $\|a\|_{L^\infty(B_0(2))}$, $\|f\|_{L^\infty(B_0(2))}$ and M such that

$$\sup_{B_0(1)} u \leq C \inf_{B_0(1)} u \ .$$

The proof follows the lines of the standard Nash-Moser iterative scheme. However, the negative power nonlinearity of u makes the proof more involved. In particular, unlike the standard Harnack inequality, Proposition 6.1 is no longer an *a priori* estimate and does not induce a control on the L^∞-norm of ∇u in $B_0(1)$.

Proof. First of all, since u, a and f are nonnegative there holds

$$\triangle_\xi u + hu \geq 0$$

in $B_0(2)$. Hence Theorem 8.18 in Gilbarg-Trudinger [22] applies and shows that for any $1 \leq p < \frac{n}{n-2}$, there exists $C_1(h,p)$ depending only on $\|h\|_{L^\infty(B_0(2))}$ and p such that

$$\inf_{B_0(1)} u \geq C_1(h,p)\|u\|_{L^p(B_0(2))}. \tag{6.1}$$

We now aim at proving that for any $p \geq 1$ there exist $C = C(a,h,f,M,p)$ such that

$$\sup_{B_0(1)} u \leq C\|u\|_{L^p(B_0(2))},$$

an estimate which together with (6.1) concludes the proof of the proposition. We adapt the steps of the proof of Theorem 4.1 in Han-Lin [23]. Let $k \geq 2^* + 2$ be given and $\eta \in C_c^\infty(B_0(2))$ be a smooth positive function with compact support in $B_0(2)$. Multiplying the equation satisfied by u by $\eta^2 u^k$ and integrating yields

$$\begin{aligned}\frac{4(k-1)}{(k+1)^2}\int_{B_0(2)}\left|\nabla u^{\frac{k+1}{2}}\right|^2\eta^2 dx \leq \int_{B_0(2)}\left(\eta^2 f u^{k+2^*-1} + a\eta^2 u^{k-2^*-1} - \eta^2 h u^{k+1}\right) dx \\ + \int_{B_0(2)} |\nabla\eta|^2 u^{k+1}\, dx\end{aligned}$$

Let $0 < r < R \leq 2$. Assume that η is compactly supported in $B_0(R)$, that it equals 1 in $B_0(r)$ and that it satisfies $|\eta| \leq 1$ and $|\nabla\eta| \leq \frac{2}{R-r}$ in $B_0(2)$. It is then

easily seen that there exists $C_2 > 0$, depending only on $\|h\|_{L^\infty(B_0(2))}$, $\|a\|_{L^\infty(B_0(2))}$, $\|f\|_{L^\infty(B_0(2))}$ and M such that

$$\int_{B_0(2)} \left|\eta^2 f u^{k+2^*-1} + a\eta^2 u^{k-2^*-1} - \eta^2 h u^{k+1}\right| dx \le C_2 \left(\int_{B_0(R)} u^{k+1} dx\right)^{\frac{k-2^*-1}{k+1}}$$

and

$$\int_{B_0(2)} |\nabla\eta|^2 u^{k+1} dx \le C_2 (R-r)^{-2} \left(\int_{B_0(R)} u^{k+1} dx\right)^{\frac{k-2^*-1}{k+1}}.$$

Independently, Sobolev's inequality shows that

$$\int_{B_0(2)} (\eta u^{\frac{k+1}{2}})^{2^*} \le K \left(\int_{B_0(R)} \left|\nabla(\eta u^{\frac{k+1}{2}})\right|^2 dx\right)^{\frac{2^*}{2}}$$

for some $K > 0$ which leads to

$$\left(\int_{B_0(2)} (\eta^{\frac{2}{k+1}} u)^{\frac{2^*}{2}(k+1)}\right)^{\frac{2}{2^*}} \le C_3 \left(\int_{B_0(R)} |\nabla\eta|^2 u^{k+1} dx + \int_{B_0(2)} \eta^2 |\nabla u^{\frac{k+1}{2}}|^2 dx\right)$$

for some positive C_3. We now let $\gamma = k+1 \ge 2^* + 3$ and $\chi = \frac{2^*}{2}$. Combining the above estimates, we obtain that

$$\|u\|_{L^{\chi\gamma}(B_0(r))} \le C_4^{\frac{1}{\gamma}} \left(\frac{\gamma}{(R-r)^2}\right)^{\frac{1}{\gamma}} \|u\|_{L^\gamma(B_0(R))}^{\frac{\gamma-2^*-2}{\gamma}} \tag{6.2}$$

for some positive constant C_4 depending only on M, $\|h\|_{L^\infty(B_0(2))}$, $\|a\|_{L^\infty(B_0(2))}$ and $\|f\|_{L^\infty(B_0(2))}$. We now pick some $0 < r < 2$ and define two sequences γ_i and r_i by $\gamma_i = \chi^i \gamma$ and $r_0 = 2, r_{i+1} = r_i - (2-r)2^{-i-1}$. Inequality (6.2) then gives that, for any $i \ge 0$:

$$\|u\|_{L^{\gamma_{i+1}}(B_0(r_{i+1}))} \le C_4^{\frac{1}{\chi^i\gamma}} \left(\frac{2\cdot 2\chi^i\gamma}{(2-r)^2}\right)^{\frac{1}{\chi^i\gamma}} \|u\|_{L^{\gamma_i}(B_0(r_i))}^{1-\frac{2^*+2}{\chi^i\gamma}}$$

and we thus obtain that there exists some constant $C_6 > 0$ depending on $\|h\|_{L^\infty(B_0(2))}$, $\|a\|_{L^\infty(B_0(2))}$, $\|f\|_{L^\infty(B_0(2))}$ and M but which does not depend on r nor on γ such that

$$\|u\|_{L^{\gamma_i}(B_0(r_i))} \le \frac{C_6}{(2-r)^{2\sum_{k=0}^{i}\frac{1}{\gamma}\chi^{-k}}} \|u\|_{L^\gamma(B_0(2))}^{\alpha_i}$$

for all $i \ge 0$, where $\alpha_i = \alpha_i(\gamma) = \prod_{k=0}^{i}\left(1 - \frac{2^*+2}{\chi^k\gamma}\right)$. Passing to the limit as $i \to \infty$ we thus obtain:

$$\|u\|_{L^\infty(B_0(r))} \le \frac{C_6}{(2-r)^{\frac{n}{\gamma}}} \|u\|_{L^\gamma(B_0(2))}^{\alpha}. \tag{6.3}$$

To conclude the proof of Proposition 6.1, we need to improve estimate (6.3). Let $1 \le p < \gamma$. There holds:

$$(2-r)^{-\frac{n}{\gamma}} \|u\|_{L^\gamma(B_0(2))}^{\alpha} \le (2-r)^{-\frac{n}{\gamma}} \|u\|_{L^\infty(B_0(2))}^{\frac{\alpha}{\gamma}(\gamma-p)} \|u\|_{L^p(B_0(2))}^{\frac{\alpha}{\gamma}},$$

so that a Young inequality of exponents $\frac{\gamma}{\gamma-p\alpha}$ and $\frac{\gamma}{p\alpha}$ combined with (6.3) yields, for any $\varepsilon > 0$:

$$\|u\|_{L^\infty(B_0(r))} \le C_6 \varepsilon^{\frac{\gamma}{\gamma-p\alpha}} \|u\|_{L^\infty(B_0(2))}^{\frac{(\gamma-p)\alpha}{\gamma-p\alpha}} + \frac{C_6}{(2-r)^{\frac{n}{p\alpha}}} \frac{p}{\gamma} \varepsilon^{-\frac{\gamma}{p\alpha}} \|u\|_{L^p(B_0(2))}. \tag{6.4}$$

It is easily seen that $\alpha(\gamma) \to 1$ as $\gamma \to \infty$. Hence, choosing $\varepsilon = (2C_6)^{-1}$ one can then pick γ large enough (depending on a, h, f and M) so as to have, in (6.4), for any $p > 1$:

$$\|u\|_{L^\infty(B_0(r))} \leq \frac{2}{3}\|u\|_{L^\infty(B_0(2))} + C_7(2-r)^{-\beta}\|u\|_{L^p(B_0(2))} ,$$

where C_7 and $\beta > 0$ only depend on $\|h\|_{L^\infty(B_0(2))}$, $\|a\|_{L^\infty(B_0(2))}$, $\|f\|_{L^\infty(B_0(2))}$, p and M. The conclusion follows using Lemma 4.3 in Han-Lin [23]. □

7. Standard elliptic theory for the Lamé operator in M

We deal in this subsection with several properties of the Lamé operator $\overrightarrow{\triangle}_g$ on (M,g). It is a differential operator between sections of the cotangent bundle T^*M. If X is a 1-form in M, $\overrightarrow{\triangle}_g$ writes in coordinates as:

$$\overrightarrow{\triangle}_g X_i = \nabla^j\nabla_j X_i + \nabla^j\nabla_i X^j - \frac{2}{n}\nabla_i(\mathrm{div}_g X).$$

If we write formally $\overrightarrow{\triangle}_g X(x) = L(x,\nabla)X$ then the principal symbol of the operator $\overrightarrow{\triangle}_g$ at some point $x \in M$ and for some $\xi \in T_xM$ is given by the determinant of the map $L(x,\xi)$ seen as a linear endomorphism of T^*_xM. Thus there holds

$$|L(x,\xi)| = \left(2 - \frac{2}{n}\right)|\xi|_g^{2n} \tag{7.1}$$

which shows that $\overrightarrow{\triangle}_g$ is uniformly elliptic in M. It also satisfies the so-called strong ellipticity condition (also called Legendre-Hadamard condition) since for any $x \in M$ and any $\eta \in T^*_xM$:

$$(L(x,\xi)\eta)_i\eta^i = |\xi|_g^2|\eta|_g^2 + \left(1 - \frac{2}{n}\right)|\langle\xi,\eta\rangle|_g^2 \geq |\xi|_g^2|\eta|_g^2 . \tag{7.2}$$

Since M is closed, integrating by parts, one gets that, for any 1-forms X and Y,

$$\int_M \langle\overrightarrow{\triangle}_g X, Y\rangle_g dv_g = \frac{1}{2}\int_M \langle\mathcal{L}_g X, \mathcal{L}_g Y\rangle_g dv_g . \tag{7.3}$$

In particular, (7.3) shows that $\overrightarrow{\triangle}_g$ is self-adjoint in $H^1(M)$ (we still denote the Sobolev space of 1-forms by $H^1(M)$ since no ambiguity will occur) and that there holds in M

$$\overrightarrow{\triangle}_g X = 0 \Longleftrightarrow \mathcal{L}_g X = 0 \tag{7.4}$$

for any 1-form X. Fields of 1-forms in M satisfying $\mathcal{L}_g X = 0$ are called conformal Killing 1-forms and by (7.3) and standard Fredholm theory the set of those 1-forms is finite dimensional. With (7.2), (7.3) and (7.4) standard results of elliptic theory for elliptic operators acting on vector bundles on closed manifolds apply, see for instance Theorem 27, Appendix H in Besse [4], or Theorem 5.20 in Giaquinta-Martinazzi [21]. In particular, for 1-forms which are L^2-orthogonal to the subspace of conformal Killing 1-forms, we have the following estimates :

Proposition 7.1. *For any $p > 1$, there exist constants $C_1 = C_1(g,p)$ and $C_2 = C_2(g,p)$ depending only on g and p such that for any 1-form X in M:*

$$\|X\|_{W^{2,p}(M)} \leq C_1\|\overrightarrow{\triangle}_g X\|_{L^p(M)} + C_2\|X\|_{L^1(M)}. \tag{7.5}$$

If, in addition, X satisfies

$$\int_M \langle X, K\rangle_g dv_g = 0 \tag{7.6}$$

for all conformal Killing 1-form K, then, for any $p > 1$, we can choose $C_2 = 0$ in (7.5).

8. Green functions for Lamé-type systems

We define, for $1 \le i \le n$, a 1-form H_i in $\mathbb{R}^n\backslash\{0\}$ by:

$$\mathcal{G}_i(y)_j = -\frac{1}{4(n-1)\omega_{n-1}}|y|^{2-n}\left((3n-2)\delta_{ij} + (n-2)\frac{y_i y_j}{|y|^2}\right) \tag{8.1}$$

for any $y \neq 0$. Note that the matrices $(\mathcal{G}_i(y)_j)_{ij}$ thus defined are symmetric: for any $y \neq 0$,

$$\mathcal{G}_i(y)_j = \mathcal{G}_j(y)_i. \tag{8.2}$$

Let X be a field of 1-forms in $\mathbb{R}^n$. Integrating by parts and using Stoke's formula it is easily seen that for any $R > 0$ and for any $x \in B_0(R)$ there holds:

$$\begin{aligned} X_i(x) = \int_{B_0(R)} \mathcal{G}_i(x-y)_j \overrightarrow{\triangle}_\xi X(y)^j dx + \int_{\partial B_0(R)} \mathcal{L}_\xi X(y)^{kl}\nu_k(y)\mathcal{G}_i(x-y)_l d\sigma \\ - \int_{\partial B_0(R)} \mathcal{L}_\xi\big(\mathcal{G}_i(x-\cdot)\big)_{kl}(y)\nu(y)^k X(y)^l d\sigma. \end{aligned} \tag{8.3}$$

This means in a distributional sense that

$$\overrightarrow{\triangle}_\xi\big(\mathcal{G}_i(x-\cdot)\big) = \delta_x e_i ,$$

where e_i is the i-th vector of the canonical basis and there holds, for any 1-form Y: $\langle \delta_x e_i, Y\rangle = Y_i(x)$. Equivalently, if we write $\mathcal{G}(x,y) = (\mathcal{G}_i(x-y)_j)_{1\le i,j\le n}$, we get that

$$\overrightarrow{\triangle}_\xi \mathcal{G}(x,\cdot) = \delta_x \mathrm{Id} , \tag{8.4}$$

where $\overrightarrow{\triangle}_\xi$ is now seen as a matrix of differential operators acting on a distribution-valued matrix. Note that the standard results of distribution theory easily extend to distribution-valued matrices, see for instance Schwartz [40].

If now X is some smooth field of 1-forms in $L^1(\mathbb{R}^n)$, the 1-form defined in $\mathbb{R}^n$ by

$$W_i(x) = \int_{\mathbb{R}^n} \mathcal{G}_i(x-y)_j Y^j(y) dy = (\mathcal{G} \star Y)_i(x)$$

satisfies in a weak sense, because of (8.3):

$$\overrightarrow{\triangle}_\xi W_i(x) = Y_i(x) . \tag{8.5}$$

The system (2.1) we are interested in in this article is invariant up to adding to W_α some conformal Killing 1-form in M. We exploit this invariance all along the article by noting that the only relevant quantity to investigate is $\mathcal{L}_g W_\alpha$ and not the 1-form W_α in itself. In particular we use several times a Green identity for $\overrightarrow{\triangle}_\xi$ with Neumann boundary conditions that is proven in what follows. We let

$$K_R = \{X \in H^1(B_0(R)), \mathcal{L}_\xi X = 0\} \tag{8.6}$$

be the Kernel subspace of 1-forms associated to the Neumann problem for $\overrightarrow{\triangle}_\xi$ in $B_0(R)$. The orthogonal subspace of K_R in $B_0(R)$ is the set of 1-forms $Y \in H^1(B_0(R))$ such that for any $K \in K_R$:

$$\int_{B_0(R)} \langle Y, K\rangle_\xi dx = 0.$$

Elements of K_R are infinitesimal generators of conformal transformations of $B_0(R)$ and are classified, see Schottenloher [39]. In particular K_R is finite dimensional, $\dim K_R = \frac{1}{2}(n+1)(n+2)$, and it is spanned by smooth 1-forms. Let $m = \frac{1}{2}(n+1)(n+2)$ and $(K_j)_{j=1\ldots m}$ be an orthonormal basis of $K_0(R)$ for the L^2-scalar product, that is

$$\int_{B_0(R)} \langle K_l, K_p\rangle_\xi dx = \delta_{lp}.$$

Given a 1-form $X \in H^1(B_0(R))$ we shall denote by $\pi_R(X)$ its orthogonal projection on K_R given by:

$$\pi_R(X) = \sum_{j=1}^m \left(\int_{B_0(R)} \langle K_j, X\rangle dx\right) K_j. \tag{8.7}$$

The following proposition states the existence of Green 1-forms satisfying Neumann boundary conditions:

Proposition 8.1. *For any $1 \le i \le n$ and any $R > 0$ there exists a unique $\mathcal{G}_{i,R}$ defined in $B_0(R) \times B_0(R)\backslash D$, where $D = \{(x,x), x \in B_0(R)\}$, such that $\mathcal{G}_{i,R}(x,\cdot)$ is orthogonal to K_R for any $x \in B_0(R)$ and such that for any smooth 1-form X in $\overline{B_0(R)}$ there holds:*

$$\begin{aligned}\big(X - \pi_R(X)\big)_i(x) = &\int_{B_0(R)} \mathcal{G}_{i,R}(x,y)_j \overrightarrow{\triangle}_\xi X(y)^j dx \\ &+ \int_{\partial B_0(R)} \mathcal{L}_\xi X(y)^{kl} \nu_k(y) \mathcal{G}_{i,R}(x,y)_l d\sigma \ ,\end{aligned} \tag{8.8}$$

where $\pi_R(X)$ is as in (8.7)*. Moreover $\mathcal{G}_{i,R}$ is continuous and continuously differentiable in each variable in $B_0(R) \times B_0(R)\backslash D$. Furthermore, if K denotes any compact set in $B_0(R)$ and if we let*

$$\delta = \frac{1}{R} d(K, \partial B_0(R)) > 0 \tag{8.9}$$

there holds:

$$|x-y||\nabla \mathcal{G}_{i,R}(x,y)| + |\mathcal{G}_{i,R}(x,y)| \le C(\delta)|x-y|^{2-n} \tag{8.10}$$

for any $x \in B_0(K)$ and any $y \in B_0(R)$, whether the derivative in (8.10) *is taken with respect to x or y, and where $C(\delta)$ is a positive constant that only depends on δ as in* (8.9) *(in particular it does not depend on x).*

Proof. The proof of this proposition goes through a sequences of claims. The techniques used are strongly inspired from Robert [37].

Claim 8.2. *Let F and G be smooth 1-forms, in $B_0(R)$ and in $\partial B_0(R)$ respectively, satisfying:*

$$\int_{B_0(R)} F_l K^l d\xi + \int_{\partial B_0(R)} G_l K^l d\sigma = 0 \tag{8.11}$$

for any $K \in K_R$, where K_R is as in (8.6). *Then there exists a unique smooth* 1*-form Z orthogonal to K_R such that*

$$\begin{cases} \overrightarrow{\triangle}_\xi Z = F & B_0(R) \\ \nu^k \mathcal{L}_\xi Z_{kl} = G_l & \partial B_0(R). \end{cases} \tag{8.12}$$

Proof. The existence and uniqueness of Z is ensured by the Lax-Milgram theorem applied on the orthogonal complement of K_R to the symmetric bilinear form

$$B(X,Y) = \frac{1}{2}\int_{B_0(R)} \langle \mathcal{L}_\xi X, \mathcal{L}_\xi Y\rangle dx$$

and to the linear form:

$$L(X) = \int_{B_0(R)} F_l X^l d\xi - \int_{\partial B_0(R)} G_l X^l d\sigma.$$

The coercivity of $B(X,X)$ on the orthogonal complement of K_R follows from the definition of K_R and is obtained via the direct method. We claim now that Z is smooth in $B_0(R)$. This is a consequence of general elliptic regularity results up to the boundary for elliptic systems satisfying complementing boundary conditions, as stated in Agmon-Douglis-Nirenberg [1]. Due to (7.1) and (7.2) the problem (8.12) is complemented so that Theorem 10.5 in Agmon-Douglis-Nirenberg [1] applies and shows that Z is smooth. ☐

For any $1 \le i \le n$ and any $x \in B_0(R)$ we let $U^R_{i,x}$ be the unique 1-form in $H^1(\overline{B_0(R)})$, orthogonal to K_R, satisfying:

$$\begin{cases} \overrightarrow{\triangle}_\xi U^R_{i,x} = -\displaystyle\sum_{j=1}^m (K_j)_i(x) K_j & B_0(R) \\ \nu^k \mathcal{L}_\xi \big(U^R_{i,x}\big)_{kl} = -\nu^k \mathcal{L}_\xi \big(H_i(x-\cdot)\big)_{kl} & \partial B_0(R). \end{cases} \tag{8.13}$$

The existence and smoothness of $U^R_{i,x}$ is ensured by Claim 8.2. Indeed, the compatibility condition (8.11) is satisfied by applying (8.3) to any $K \in K_R$.

We now let, for $x \neq y$:

$$\mathcal{G}_{i,R}(x,y) = \mathcal{G}_i(x-y) + U^R_{i,x}(y) - \sum_{j=1}^m \left(\int_{B_0(R)} \langle K_j, \mathcal{G}_i(x-\cdot)\rangle dy\right) K_j(y), \tag{8.14}$$

where $\mathcal{G}_i$ is as in (8.1). By construction, $\mathcal{G}_{i,R}(x,\cdot)$ is a 1-form defined in $B_0(R)\backslash\{x\}$. It clearly belongs to $L^2(B_0(R))$, is orthogonal to K_R and continuously differentiable in $B_0(R)\backslash\{x\}$. Combining (8.3) and (8.13) it is easily seen that (8.8) holds. The next three claims aim at finishing the proof of the proposition. The first one is a uniqueness result.

Claim 8.3. *Assume that for some $1 \le i \le \frac{n}{n-2}$ and for some $x \in B_0(R)$ there exists a* 1*-form M_i in $L^1(B_0(R))$ such that for any $X \in C^2(\overline{B_0(R)})$ satisfying that $\mathcal{L}_\xi X_{kl}\nu^k = 0$ on $\partial B_0(R)$ there holds:*

$$\int_{B_0(R)} \langle M_i, \overrightarrow{\triangle}_\xi X\rangle_\xi d\xi = \left(X - \pi(X)\right)_i(x). \tag{8.15}$$

Then $\mathcal{G}_{i,R}(x,\cdot) - M_i \in K_R$, where K_R is as in (8.6).

Proof. Let $F_i = \mathcal{G}_{i,R}(x,\cdot) - M_i$. Let Y be a smooth 1-form with compact support in $B_0(R)$. By Claim 8.2 there exists a smooth 1-form X in $B_0(R)$ such that $\overrightarrow{\triangle}_\xi X = Y - \pi_R(Y)$ in $B_0(R)$ and $\mathcal{L}_\xi X_{kl}\nu^k = 0$ in $\partial B_0(R)$, where π_R is as in (8.7). Using (8.8) and (8.15) there holds:

$$0 = \int_{B_0(R)} \langle F_i, \overrightarrow{\triangle}_\xi X\rangle_\xi d\xi = \int_{B_0(R)} \langle F_i, Y - \pi_R(Y)\rangle_\xi d\xi = \int_{B_0(R)} \langle F_i - \pi_R(F_i), Y\rangle_\xi d\xi$$

by definition of π_R. Assume for a while that F_i belongs to $L^p(B_0(R))$ for some $p > 1$. A density argument then shows that $F_i = \pi_R(F_i)$.
It thus remains to prove that $F_i \in L^p(B_0(R))$ for some $p > 1$. We only need to prove this for M_i. Let $p \in (1, \frac{n}{n-2})$ and define $q = \frac{p}{p-1}$. Let Y be a smooth 1-form compactly supported in $B_0(R)$. By Claim 8.2 there exists a smooth 1-form X in $B_0(R)$, orthogonal to K_R, such that $\overrightarrow{\triangle}_\xi X = Y - \pi_R(Y)$ in $B_0(R)$ and $\mathcal{L}_\xi X_{kl}\nu^k = 0$ in $\partial B_0(R)$. Then the definition of π_R and (8.15) yield:

$$\begin{aligned}\int_{B_0(R)} \langle M_i - \pi_R(M_i), Y\rangle_\xi d\xi &= \int_{B_0(R)} \langle M_i, Y - \pi_R(Y)\rangle_\xi d\xi \\ &= \int_{B_0(R)} \langle M_i, \overrightarrow{\triangle}_\xi X\rangle_\xi d\xi \\ &= X_i(x).\end{aligned}$$

Elliptic regularity results for complemented elliptic systems, as those stated in the proof of Claim 8.2, show that there exists a constant C only depending on q such that $\|X\|_{W^{2,q}} \le C\|Y - \pi_R(Y)\|_{L^q}$ where we omit to say that these norms are taken on $B_0(R)$ for the sake of clarity. Since $q > \frac{n}{2}$ we thus obtain, using the Sobolev inequality for the embedding of $W^{2,q}$ in $C^0(\overline{B_0(R)})$:

$$\left| \int_{B_0(R)} \langle M_i - \pi_R(M_i), Y\rangle_\xi d\xi \right| \le C\|Y - \pi_R(Y)\|_{L^q} \le C\|Y\|_{L^q}.$$

A density argument then shows that $M_i - \pi_R(M_i)$ and thus M_i belongs to $L^p(B_0(R))$ for all $p \in (1, \frac{n}{n-2})$. ☐

We now state some rescaling-invariance property of the Green 1-forms $\mathcal{G}_{i,R}$:

Claim 8.4. *For any $R > 0$ there holds:*

$$\mathcal{G}_{i,R}(x,y) = \frac{1}{R}\mathcal{G}_{i,1}\left(\frac{x}{R}, \frac{y}{R}\right) \tag{8.16}$$

for any $x, y \in B_0(R)$.

Proof. Let Y be a smooth, compactly supported 1-form in $B_0(R)$. We define, in $B_0(1)$, $Y_R = Y(R\cdot)$, which is then compactly supported in $B_0(1)$. Let $x \in B_0(R)$ and $1 \le i \le n$. Equation (8.8) shows that

$$\left(Y_R - \pi_1(Y_R)\right)_i \left(\frac{x}{R}\right) = \int_{B_0(1)} \left\langle \mathcal{G}_{i,1}\left(\frac{x}{R}, y\right), \overrightarrow{\triangle}_\xi Y_R(y)\right\rangle_\xi d\xi. \tag{8.17}$$

Since for any $y \in B_0(1)$ there holds $\overrightarrow{\triangle}_\xi Y_R(y) = R^2 \overrightarrow{\triangle}_\xi Y(Ry)$ we easily obtain:

$$\int_{B_0(1)} \left\langle \mathcal{G}_{i,1}\left(\frac{x}{R}, y\right), \overrightarrow{\triangle}_\xi Y_R(y)\right\rangle_\xi dy = \frac{1}{R}\int_{B_0(R)} \left\langle \mathcal{G}_{i,1}\left(\frac{x}{R}, \frac{y}{R}\right), \overrightarrow{\triangle}_\xi Y(y)\right\rangle_\xi dy.$$

Let now $(L_j)_{1\le j\le m}$ be an orthonormal basis for K_1, where K_1 is as in (8.6). Let $Z_j = R^{-\frac{n}{2}} L_j\left(\frac{x}{R}\right)$ for any $1\le j\le m$ and any $x\in B_0(R)$. Then $(Z_j)_{1\le j\le m}$ is an orthonormal basis for K_R since there holds

$$\int_{B_0(R)} \langle Z_k, Z_l\rangle_\xi d\xi = \int_{B_0(R)} R^{-n}\langle L_k\left(\frac{y}{R}\right), L_l\left(\frac{y}{R}\right)\rangle_\xi dy = \int_{B_0(1)} \langle L_k, L_l\rangle_\xi d\xi = \delta_{kl}.$$

Hence one has, by definition of π_R:

$$\begin{aligned}\pi_R(Y)(x) &= \sum_{j=1}^m \left(\int_{B_0(R)} \langle R^{-\frac{n}{2}} L_j\left(\frac{y}{R}\right), Y(y)\rangle dy\right) R^{-\frac{n}{2}} L_j\left(\frac{x}{R}\right)\\ &= \sum_{j=1}^m \left(\int_{B_0(1)} \langle L_j(y), Y_R(y)\rangle dy\right) L_j\left(\frac{x}{R}\right) = \pi_1(Y_R)\left(\frac{x}{R}\right).\end{aligned}$$

In the end (8.17) becomes:

$$\left(Y-\pi_R(Y)\right)_i(x) = \int_{B_0(R)} \left\langle \frac{1}{R}\mathcal{G}_{i,1}\left(\frac{x}{R},\frac{y}{R}\right), \overrightarrow{\triangle}_\xi Y(y)\right\rangle_\xi dy.$$

Finally, $\frac{1}{R}\mathcal{G}_{i,1}\left(\frac{x}{R},\frac{\cdot}{R}\right)$ is orthogonal to K_R: indeed for $1\le j\le m$ there holds:

$$\begin{aligned}\int_{B_0(R)} \left\langle \frac{1}{R}\mathcal{G}_{i,1}\left(\frac{x}{R},\frac{y}{R}\right), R^{-\frac{n}{2}} L_j\left(\frac{y}{R}\right)\right\rangle_\xi dy &= R^{\frac{1}{2}}\int_{B_0(1)} \langle \mathcal{G}_{i,1}(\frac{x}{R}, y), L_j(y)\rangle_\xi dy\\ &= 0\,,\end{aligned}$$

where the last equality is true since $\mathcal{G}_{i,1}(\frac{x}{R},\cdot)$ is orthogonal to K_1 by definition. Using Claim 8.3 we then obtain (8.16). □

The last ingredient of the proof is a symmetry property of $\mathcal{G}_{i,R}$:

Claim 8.5. *For any $x,y\in B_0(R)$ there holds:*

$$\mathcal{G}_{i,R}(x,y)_j = \mathcal{G}_{j,R}(y,x)_i - \pi_R\left({}^t\mathcal{G}_i(\cdot,x)\right)_j \tag{8.18}$$

where we have set:

$${}^t\mathcal{G}_i(\cdot,x)_j(y) = \mathcal{G}_{j,R}(y,x)_i. \tag{8.19}$$

Proof. Let $\Psi\in C^0(\overline{B_0(R)})$ be a 1-form orthogonal to K_R. We define a 1-form in $B_0(R)$ by

$$H_i(x) = \int_{B_0(R)} \mathcal{G}_{j,R}(y,x)_i \Psi(y)^j dy\,. \tag{8.20}$$

By the explicit construction of $\mathcal{G}_{j,R}$ in (8.14) it is easily seen that H is continuous in $\overline{B_0(R)}$. Also, H is orthogonal to the conformal Killing 1-forms since by Fubini's theorem, for any $K\in K_R$,

$$\int_{B_0(R)} H_i(y)K^i(y)dy = \int_{B_0(R)} \Psi^j(z)\int_{B_0(R)} \mathcal{G}_{j,R}(z,y)_i K^i(y) dy dz = 0 \tag{8.21}$$

since by construction $\mathcal{G}_{j,R}(z,\cdot)$ is orthogonal to K_R for any $z\in B_0(R)$. By Claim 8.2 and since Ψ is orthogonal to K_R we can let F be the unique C^1 1-form in $B_0(R)$ orthogonal to K_R satisfying $\overrightarrow{\triangle}_\xi F = \Psi$ in $B_0(R)$ and $\mathcal{L}_\xi F_{kl}\nu^k = 0$ in $\partial B_0(R)$. Let

also Φ be a smooth 1-form such that $\mathcal{L}_\xi \Phi_{kl} \nu^k = 0$ on $\partial B_0(R)$. With Fubini's theorem, equation (8.8) and using the properties of Ψ and Φ there holds:

$$\begin{aligned}
\int_{B_0(R)} H_i(y) \overrightarrow{\triangle}_\xi \Phi^i(y) dy &= \int_{B_0(R)} \Psi^j(z) \int_{B_0(R)} \mathcal{G}_{j,R}(z,y)_i \overrightarrow{\triangle}_\xi \Phi^i(y) dy dz \\
&= \int_{B_0(R)} \Psi^j(z) \left(\Phi - \pi_R(\Phi)\right)_j(z) dz \\
&= \int_{B_0(R)} \Psi^j(z) \Phi_j(z) dz \\
&= \int_{B_0(R)} \Phi_j(z) \overrightarrow{\triangle}_\xi F^j(z) dz \\
&= \int_{B_0(R)} F_j(z) \overrightarrow{\triangle}_\xi \Phi^j(z) dz \ ,
\end{aligned}$$

where we integrated by parts to obtain the last inequality since the boundary terms vanish. In particular:

$$\int_{B_0(R)} (H - F)_j (y) \overrightarrow{\triangle}_\xi \Phi^j(y) dy = 0$$

for any smooth Φ with $\mathcal{L}_\xi \Phi_{kl} \nu^k = 0$ on $\partial B_0(R)$. Note that by a density argument the above inequality remains true for $\Phi \in W^{2,p}(B_0(R))$ for any $p > 1$ and orthogonal to K_R. By construction F is orthogonal to K_R and thanks to (8.21) so is H. By Claim 8.2 we can thus choose Φ to be the unique 1-form orthogonal to K_R satisfying $\overrightarrow{\triangle}_\xi \Phi = F - H$ in $B_0(R)$ and $\mathcal{L}_\xi \Phi_{kl} \nu^k = 0$ in $\partial B_0(R)$ to obtain, with the above inequality, that $F = H$. Independently, using (8.8) gives:

$$F_i(x) = \int_{B_0(R)} \mathcal{G}_{i,R}(x,y)_j \Psi^j(y) dy$$

so that

$$\int_{B_0(R)} \left(\mathcal{G}_{j,R}(y,x)_i - \mathcal{G}_{i,R}(x,y)_j\right) \Psi^j(y) dy = 0 \tag{8.22}$$

for any continuous Killing-free Ψ. Let now X be any smooth 1-form in $B_0(R)$. Choose $\Psi = X - \pi_R(X)$. There holds

$$\begin{aligned}
&\int_{B_0(R)} \mathcal{G}_{j,R}(y,x)_i \pi_R(X)^j(y) dy \\
&\quad = \sum_{p=1}^m \int_{B_0(R)} \mathcal{G}_{j,R}(y,x)_i \left(\int_{B_0(R)} \langle K_p(z), X(z) \rangle_\xi dz \right) K_p(y)^j dy \\
&\quad = \sum_{p=1}^m \int_{B_0(R)} X_l(z) \left(\int_{B_0(R)} \mathcal{G}_{j,R}(y,x)_i K_p(y)^j dy \right) K_p^l(z) dz \\
&\quad = \sum_{p=1}^m \int_{B_0(R)} X_l(z) \pi_R \left({}^t\mathcal{G}_i(\cdot,x)\right)^l(z) dz \ ,
\end{aligned}$$

where ${}^t\mathcal{G}_i(\cdot,x)$ is as in (8.19). Since $\mathcal{G}_{i,R}(x,\cdot)$ has no conformal Killing part, equation (8.22) becomes:

$$\int_{B_0(R)} \left(\mathcal{G}_{j,R}(y,x)_i - \mathcal{G}_{i,R}(x,y)_j - \pi_R\left({}^t\mathcal{G}_i(\cdot,x)\right)_j(y) \right) X^j(y) dy = 0$$

for any smooth 1-form X and this concludes the proof of the claim. $\square$

We are now able to end the proof of Proposition 8.1. Let $x \in B_0(1)$ and consider $U^1_{i,x}$ as defined in (8.13). Since $\overrightarrow{\triangle}_\xi$ is coercive on the orthogonal of K_1 we can use elliptic regularity results – as those stated in the proof of Claim 8.2 – to get that there exist positive constants C_1, C_2 that do not depend on x such that

$$\|U^1_{i,x}\|_{C^1(\overline{B_0(R)})} \le C_1 + C_2 \|\mathcal{L}_\xi \mathcal{G}_i(x - \cdot), \nu \otimes \cdot\|_{C^1(\partial B_0(R))} \ , \tag{8.23}$$

where we have let $(\mathcal{L}_\xi \mathcal{G}_i(x-\cdot), \nu \otimes \cdot)_l = \nu^k \mathcal{L}_\xi \big(\mathcal{G}_i(x-\cdot)\big)_{kl}$. Let K be some compact set in $B_0(1)$ and assume $x \in K$. It is easily seen by the definition of $\mathcal{G}_i$ as in (8.1) that

$$\|\mathcal{L}_\xi \mathcal{G}_i(x - \cdot), \nu \otimes \cdot\|_{C^1(\partial B_0(R))} \le C_3 d\left(K, \partial B_0(1)\right)^{1-n}$$

for some positive constant C_3 independent of x. By the definition of $\mathcal{G}_{i,1}(x,\cdot)$ in (8.14) one therefore easily obtains that for $x \in K$ and $y \in B_0(1)$:

$$|x - y||\nabla_y \mathcal{G}_{i,1}(x,y)| + |\mathcal{G}_{i,1}(x,y)| \le C(\delta)|x-y|^{2-n} \ , \tag{8.24}$$

where δ is as in (8.9). This gives (8.10) when the derivative is taken with respect to y. The same estimate when the derivative is taken with respect to x is obtained differentiating (8.18) and combining with (8.24). Finally, (8.10) for any positive R is obtained combining (8.24) with Claim 8.4. $\square$

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